Chapter 4

Radicals and surds

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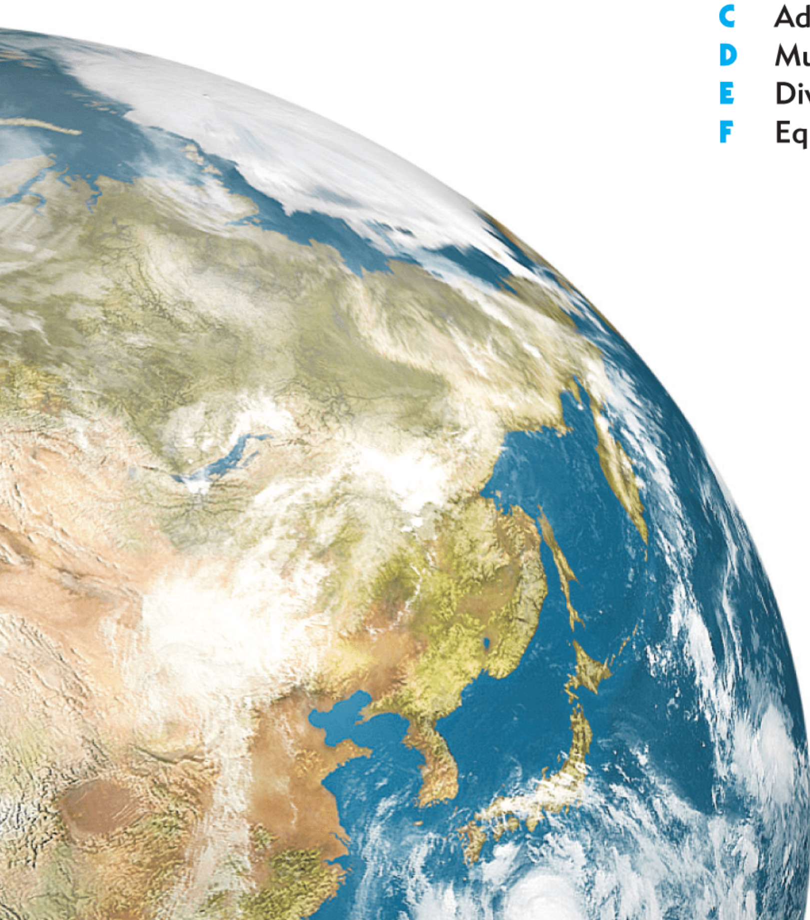
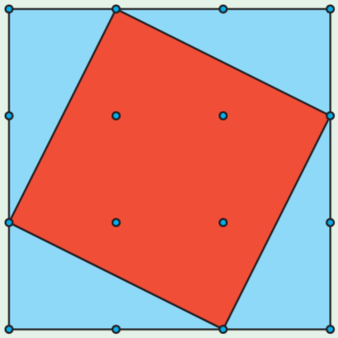


OPENING PROBLEM

Pamela's students claim that since an irrational number cannot be written as a fraction, it cannot be placed on a number line.

To demonstrate that an irrational length can be represented, and therefore placed on a number line, Pamela draws this figure.



Things to think about:

a Find:
  i the area of the large square
  ii the total area of the blue triangles.

b Using your answers to **a**, find the area of the red square.

c Hence explain why the side length of the red square is $\sqrt{5}$ units.

d What is the perimeter of the red square?

A RADICALS

In **Chapter 1** we encountered values such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt[3]{8}$. These numbers are known as *radicals*.

A **radical** is a number that is written using the radical sign $\sqrt{}$.

Radicals occur frequently in mathematics, often as solutions to equations involving squared terms. We will see a typical example of this in **Chapter 5** when we study Pythagoras' theorem.

SQUARE ROOTS

The **square root of** $\boldsymbol{a}$, written $\sqrt{a}$, is the *positive* solution of the equation $x^2 = a$.

$\sqrt{a}$ is the *positive* number which obeys the rule $\sqrt{a} \times \sqrt{a} = a$.

For example, $\sqrt{2} \times \sqrt{2} = 2$, $\sqrt{3} \times \sqrt{3} = 3$, $\sqrt{4} \times \sqrt{4} = 4$, and so on.

We know that the square of any real number is non-negative. This means that:

$\sqrt{a}$ is real only if $a \geqslant 0$.

HIGHER ROOTS

In this course we will concentrate mainly on square roots, but it is important to understand that other radicals exist. For example:

The **cube root of** a, written $\sqrt[3]{a}$, satisfies the rule $\left(\sqrt[3]{a}\right)^3 = a$.

If $a > 0$ then $\sqrt[3]{a} > 0$.

If $a < 0$ then $\sqrt[3]{a} < 0$.

We can define higher roots in a similar way.

RATIONAL AND IRRATIONAL RADICALS

In **Chapter 2** we saw that the set of **real numbers** $\mathbb{R}$ can be divided into the set of **rational numbers** $\mathbb{Q}$, and the set of **irrational numbers** $\mathbb{Q}'$.

Remember that:

An **irrational number** is a real number which cannot be written in the form $\dfrac{p}{q}$, where p and q are integers, $q \neq 0$.

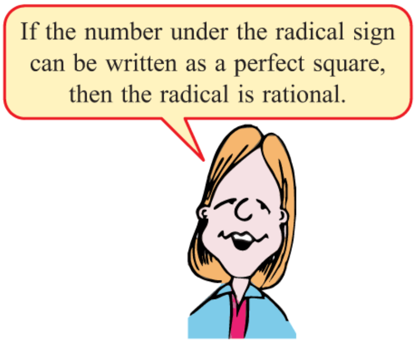
Radical numbers may be rational or irrational. An irrational radical is called a surd.

Examples of rational radicals include:

$$\sqrt{4} = 2 = \tfrac{2}{1}$$

$$\sqrt{\tfrac{9}{16}} = \sqrt{\left(\tfrac{3}{4}\right)^2} = \tfrac{3}{4}$$

Two examples of surds are $\sqrt{2} \approx 1.414\,214$
and $\sqrt{3} \approx 1.732\,051$.



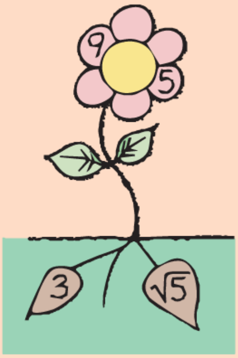
HISTORICAL NOTE

When the Golden Age of the Greeks was past, the writings of the Greeks were preserved, translated into Arabic and extended by Arabic mathematicians in the regions currently known as Iraq and Iran and also in Moslem Spain. The word *surd* came about because of an error in translation.

The Greek word *alogos* meaning 'irrational', or 'without reason', was translated as the Arabic word *asamm* which means 'irrational', but also means 'deaf'. Thus rational and irrational numbers were called 'audible' and 'inaudible' numbers respectively by the Arabic mathematician **Al-Khwarizmi**, from Persia, around 825 AD.

This later led to the Arabic word *asamm* meaning 'deaf' or 'dumb' for irrational numbers being translated into Latin as *surdus* meaning 'deaf' or 'mute'. The European mathematician, Gherardo of Cremona (c. 1150), adopted the word surd.

The origin of the root symbol $\sqrt{\ }$ is not clear. Some sources suggest that the symbol was first used by Arabic mathematicians. It is believed that the modern square root symbol developed from the letter r which is the first letter in the Latin word radix, meaning 'root', from which we get the word *radical*.



SIMPLIFYING RADICALS

In previous years we have established that:

- $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ for $a \geqslant 0,\ b \geqslant 0$
- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0,\ b > 0$

Example 1 Self Tutor

Simplify:

a $(\sqrt{5})^2$

b $\left(\frac{1}{\sqrt{5}}\right)^2$

a $(\sqrt{5})^2$
$= \sqrt{5} \times \sqrt{5}$
$= 5$

b $\left(\frac{1}{\sqrt{5}}\right)^2$
$= \frac{1}{\sqrt{5}} \times \frac{1}{\sqrt{5}}$
$= \frac{1}{5}$

Example 2 Self Tutor

Simplify:

a $(2\sqrt{5})^3$

b $-2\sqrt{5} \times 3\sqrt{5}$

a $(2\sqrt{5})^3$
$= 2\sqrt{5} \times 2\sqrt{5} \times 2\sqrt{5}$
$= 2 \times 2 \times 2 \times \sqrt{5} \times \sqrt{5} \times \sqrt{5}$
$= 8 \times 5 \times \sqrt{5}$
$= 40\sqrt{5}$

b $-2\sqrt{5} \times 3\sqrt{5}$
$= -2 \times 3 \times \sqrt{5} \times \sqrt{5}$
$= -6 \times 5$
$= -30$

EXERCISE 4A

1 Simplify:

a $(\sqrt{7})^2$ **b** $(\sqrt{13})^2$ **c** $(\sqrt{15})^2$ **d** $(\sqrt{24})^2$

e $(\sqrt{2})^3$ **f** $(\sqrt{2})^4$ **g** $(\sqrt{2})^5$ **h** $\left(\frac{1}{\sqrt{2}}\right)^2$

i $\left(\frac{1}{\sqrt{3}}\right)^2$ **j** $\left(\frac{2}{\sqrt{11}}\right)^2$ **k** $\left(\frac{\sqrt{3}}{\sqrt{17}}\right)^2$ **l** $\left(\sqrt{\frac{2}{23}}\right)^2$

2 Simplify:

a $(\sqrt[3]{2})^3$ **b** $(\sqrt[3]{-5})^3$ **c** $\left(\frac{1}{\sqrt[3]{5}}\right)^3$

3 Simplify:

a $4\sqrt{3} \times \sqrt{3}$ **b** $3\sqrt{2} \times \sqrt{2}$ **c** $\sqrt{5} \times 6\sqrt{5}$

d $3\sqrt{2} \times 4\sqrt{2}$ **e** $-2\sqrt{3} \times 5\sqrt{3}$ **f** $3\sqrt{5} \times (-2\sqrt{5})$

g $-2\sqrt{2} \times (-3\sqrt{2})$ **h** $(3\sqrt{2})^2$ **i** $(3\sqrt{2})^3$

j $(4\sqrt{3})^2$ **k** $(4\sqrt{3})^3$ **l** $(2\sqrt{2})^4$

m $\sqrt{5} \times (3\sqrt{5})^2$ **n** $-2\sqrt{3} \times (5\sqrt{3})^2$ **o** $(2\sqrt{2})^3 \times (-7\sqrt{2})$

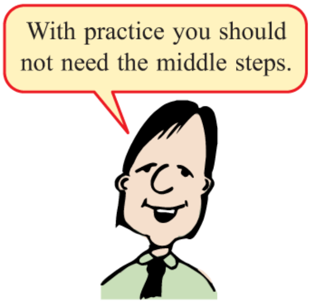
Example 3 **Self Tutor**

Write in simplest form:

a $\sqrt{3} \times \sqrt{2}$ **b** $2\sqrt{5} \times 3\sqrt{2}$

a $\sqrt{3} \times \sqrt{2}$
$= \sqrt{3 \times 2}$
$= \sqrt{6}$

b $2\sqrt{5} \times 3\sqrt{2}$
$= 2 \times 3 \times \sqrt{5} \times \sqrt{2}$
$= 6 \times \sqrt{5 \times 2}$
$= 6\sqrt{10}$



4 Simplify:

a $\sqrt{2} \times \sqrt{5}$ **b** $\sqrt{3} \times \sqrt{7}$ **c** $\sqrt{3} \times \sqrt{11}$
d $\sqrt{7} \times \sqrt{7}$ **e** $\sqrt{3} \times 2\sqrt{3}$ **f** $2\sqrt{2} \times (-\sqrt{5})$
g $3\sqrt{3} \times 2\sqrt{2}$ **h** $2\sqrt{3} \times 3\sqrt{5}$ **i** $\sqrt{2} \times \sqrt{3} \times \sqrt{5}$
j $\sqrt{3} \times \sqrt{2} \times 2\sqrt{2}$ **k** $-3\sqrt{2} \times (\sqrt{2})^3$ **l** $(3\sqrt{2})^3 \times (\sqrt{3})^3$

Example 4 **Self Tutor**

Simplify: **a** $\dfrac{\sqrt{32}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{12}}{2\sqrt{3}}$

a $\dfrac{\sqrt{32}}{\sqrt{2}}$
$= \sqrt{\frac{32}{2}}$
$= \sqrt{16}$
$= 4$

b $\dfrac{\sqrt{12}}{2\sqrt{3}}$
$= \frac{1}{2}\sqrt{\frac{12}{3}}$ $\left\{\text{using } \dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}\right\}$
$= \frac{1}{2}\sqrt{4}$
$= \frac{1}{2} \times 2$
$= 1$

5 Simplify:

a $\dfrac{\sqrt{8}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{2}}{\sqrt{8}}$ **c** $\dfrac{\sqrt{18}}{\sqrt{2}}$ **d** $\dfrac{\sqrt{2}}{\sqrt{18}}$
e $\dfrac{\sqrt{20}}{\sqrt{5}}$ **f** $\dfrac{\sqrt{5}}{\sqrt{20}}$ **g** $\dfrac{\sqrt{27}}{\sqrt{3}}$ **h** $\dfrac{\sqrt{18}}{\sqrt{3}}$
i $\dfrac{\sqrt{3}}{\sqrt{30}}$ **j** $\dfrac{\sqrt{50}}{\sqrt{2}}$ **k** $\dfrac{2\sqrt{6}}{\sqrt{24}}$ **l** $\dfrac{5\sqrt{75}}{\sqrt{3}}$

6 Simplify:

a $\sqrt{\frac{1}{25}}$ **b** $\sqrt{\frac{16}{9}}$ **c** $\sqrt{3\frac{1}{16}}$ **d** $\sqrt{20\frac{1}{4}}$

7 **a** Prove that $\sqrt{a}\sqrt{b} = \sqrt{ab}$ for all positive numbers a and b.
Hint: Consider $(\sqrt{a}\sqrt{b})^2$ and $(\sqrt{ab})^2$.

b Prove that $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ for $a \geqslant 0$ and $b > 0$.

8 **a** Is $\sqrt{9}+\sqrt{16}=\sqrt{9+16}$? Is $\sqrt{25}-\sqrt{16}=\sqrt{25-16}$?

b Are $\sqrt{a}+\sqrt{b}=\sqrt{a+b}$ and $\sqrt{a}-\sqrt{b}=\sqrt{a-b}$ possible laws for radicals?

B SIMPLEST RADICAL FORM

A radical is in **simplest form** when the number under the radical sign is the smallest possible integer.

Example 5 Self Tutor

Write $\sqrt{28}$ in simplest radical form.

$$\begin{aligned} &\sqrt{28} \\ &= \sqrt{4\times 7} \\ &= \sqrt{4}\times\sqrt{7} \\ &= 2\sqrt{7} \end{aligned}$$

EXERCISE 4B

1 Write in simplest radical form:

a $\sqrt{24}$ **b** $\sqrt{50}$ **c** $\sqrt{54}$ **d** $\sqrt{40}$

e $\sqrt{56}$ **f** $\sqrt{63}$ **g** $\sqrt{52}$ **h** $\sqrt{44}$

i $\sqrt{60}$ **j** $\sqrt{90}$ **k** $\sqrt{96}$ **l** $\sqrt{68}$

m $\sqrt{175}$ **n** $\sqrt{162}$ **o** $\sqrt{128}$ **p** $\sqrt{700}$

2 Write in simplest radical form:

a $\sqrt{\frac{5}{9}}$ **b** $\sqrt{\frac{18}{4}}$ **c** $\sqrt{\frac{12}{16}}$ **d** $\sqrt{\frac{75}{36}}$

3 Write in simplest radical form $a+b\sqrt{n}$ where $a, b\in\mathbb{Q}$, $n\in\mathbb{Z}$:

a $\dfrac{4+\sqrt{8}}{2}$ **b** $\dfrac{6-\sqrt{12}}{2}$ **c** $\dfrac{4+\sqrt{20}}{4}$ **d** $\dfrac{8-\sqrt{32}}{4}$

e $\dfrac{12+\sqrt{72}}{6}$ **f** $\dfrac{18+\sqrt{27}}{6}$ **g** $\dfrac{14-\sqrt{60}}{8}$ **h** $\dfrac{5-\sqrt{200}}{10}$

C ADDING AND SUBTRACTING RADICALS

We can add and subtract 'like' radicals in the same way as we do 'like' terms in algebra.

For example:

- just as $3a+2a=5a$, $3\sqrt{2}+2\sqrt{2}=5\sqrt{2}$
- just as $6b-4b=2b$, $6\sqrt{3}-4\sqrt{3}=2\sqrt{3}$.

Example 6 Self Tutor

Simplify:

a $3\sqrt{2}-4\sqrt{2}$ **b** $5\sqrt{3}+2\sqrt{5}-\sqrt{3}+7\sqrt{5}$

a
$$\begin{aligned}&3\sqrt{2}-4\sqrt{2}\\&=-1\sqrt{2}\\&=-\sqrt{2}\end{aligned}$$

b

'like' radicals

$$5\sqrt{3}+2\sqrt{5}-\sqrt{3}+7\sqrt{5}$$

'like' radicals

$$=4\sqrt{3}+9\sqrt{5}$$

We write the whole number first, then the radical. So, we write $4\sqrt{3}$ not $\sqrt{3}\,4$.

EXERCISE 4C

1 Simplify:

a $3\sqrt{2}+7\sqrt{2}$ **b** $11\sqrt{3}-8\sqrt{3}$ **c** $6\sqrt{5}-7\sqrt{5}$

d $-\sqrt{10}+2\sqrt{10}$ **e** $\sqrt{6}+7\sqrt{6}-3\sqrt{6}$ **f** $9\sqrt{15}-4\sqrt{15}-11\sqrt{15}$

2 Simplify:

a $5\sqrt{2}-\sqrt{3}+\sqrt{2}-\sqrt{3}$ **b** $4\sqrt{7}+5\sqrt{6}+3\sqrt{7}-2\sqrt{6}$

c $9\sqrt{10}-5\sqrt{5}+8\sqrt{5}-\sqrt{10}$ **d** $11\sqrt{3}-8\sqrt{13}+\sqrt{13}-13\sqrt{3}$

e $\sqrt{7}+5\sqrt{11}+9\sqrt{7}-4\sqrt{7}$ **f** $-6\sqrt{14}-3\sqrt{14}-2\sqrt{6}+10\sqrt{14}$

g $14+6\sqrt{17}-8\sqrt{17}-3$ **h** $-8-\sqrt{15}+7\sqrt{15}-13$

Example 7 Self Tutor

Simplify: $\dfrac{\sqrt{7}}{2}+\dfrac{\sqrt{7}}{3}$

$$\begin{aligned}&\frac{\sqrt{7}}{2}+\frac{\sqrt{7}}{3}\qquad\{\text{LCD}=6\}\\&=\frac{\sqrt{7}\times 3}{2\times 3}+\frac{\sqrt{7}\times 2}{3\times 2}\\&=\frac{3\sqrt{7}}{6}+\frac{2\sqrt{7}}{6}\\&=\frac{3\sqrt{7}+2\sqrt{7}}{6}\\&=\frac{5\sqrt{7}}{6}\end{aligned}$$

3 Simplify:

a $\dfrac{\sqrt{5}}{3}+\dfrac{\sqrt{5}}{4}$ **b** $\dfrac{\sqrt{6}}{2}-\dfrac{\sqrt{6}}{7}$ **c** $\dfrac{5\sqrt{3}}{6}+\dfrac{\sqrt{3}}{8}$

d $\dfrac{2\sqrt{11}}{9}-\dfrac{8\sqrt{11}}{15}$ **e** $\dfrac{\sqrt{10}}{2}+\dfrac{\sqrt{10}}{3}+\dfrac{\sqrt{10}}{4}$ **f** $\sqrt{2}+\dfrac{5\sqrt{2}}{14}-\dfrac{7\sqrt{2}}{4}$

4 Show that:

a $\sqrt{20}+\sqrt{5}=\sqrt{45}$ **b** $\sqrt{147}-\sqrt{75}=\sqrt{12}$

5 Answer the **Opening Problem** on page **64**.

D MULTIPLICATIONS INVOLVING RADICALS

The rules for expanding expressions containing radicals are identical to those for ordinary algebra.

$$\begin{aligned} a(b+c) &= ab+ac \\ (a+b)(c+d) &= ac+ad+bc+bd \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a+b)(a-b) &= a^2-b^2 \end{aligned}$$

Example 8 Self Tutor

Expand and simplify:

a $\sqrt{2}(\sqrt{2}+\sqrt{3})$ **b** $\sqrt{3}(6-2\sqrt{3})$

a $\sqrt{2}(\sqrt{2}+\sqrt{3})$
$= \sqrt{2}\times\sqrt{2} + \sqrt{2}\times\sqrt{3}$
$= 2+\sqrt{6}$

b $\sqrt{3}(6-2\sqrt{3})$
$= \sqrt{3}\times 6+\sqrt{3}\times(-2\sqrt{3})$
$= 6\sqrt{3}-6$

EXERCISE 4D

1 Expand and simplify:

a $\sqrt{2}(\sqrt{5}+\sqrt{2})$ **b** $\sqrt{2}(3-\sqrt{2})$ **c** $\sqrt{3}(\sqrt{3}+1)$
d $\sqrt{3}(1-\sqrt{3})$ **e** $\sqrt{7}(7-\sqrt{7})$ **f** $\sqrt{5}(2-\sqrt{5})$
g $\sqrt{11}(2\sqrt{11}-1)$ **h** $\sqrt{6}(1-2\sqrt{6})$ **i** $\sqrt{3}(\sqrt{3}+\sqrt{2}-1)$
j $2\sqrt{3}(\sqrt{3}-\sqrt{5})$ **k** $2\sqrt{5}(3-\sqrt{5})$ **l** $3\sqrt{5}(2\sqrt{5}+\sqrt{2})$

Example 9 Self Tutor

Expand and simplify:

a $-\sqrt{2}(\sqrt{2}+3)$ **b** $-\sqrt{3}(7-2\sqrt{3})$

a $-\sqrt{2}(\sqrt{2}+3)$
$= -\sqrt{2}\times\sqrt{2} + -\sqrt{2}\times 3$
$= -2-3\sqrt{2}$

b $-\sqrt{3}(7-2\sqrt{3})$
$= -\sqrt{3}\times 7+-\sqrt{3}\times(-2\sqrt{3})$
$= -7\sqrt{3}+6$

2 Expand and simplify:

a $-\sqrt{2}(3-\sqrt{2})$ **b** $-\sqrt{2}(\sqrt{2}+\sqrt{3})$ **c** $-\sqrt{2}(4-\sqrt{2})$
d $-\sqrt{3}(1+\sqrt{3})$ **e** $-\sqrt{3}(\sqrt{3}+2)$ **f** $-\sqrt{5}(2+\sqrt{5})$
g $-(\sqrt{2}+3)$ **h** $-\sqrt{5}(\sqrt{5}-4)$ **i** $-(3-\sqrt{7})$
j $-\sqrt{11}(2-\sqrt{11})$ **k** $-(\sqrt{3}-\sqrt{7})$ **l** $-2\sqrt{2}(1-\sqrt{2})$
m $-3\sqrt{3}(5-\sqrt{3})$ **n** $-7\sqrt{2}(\sqrt{2}+\sqrt{6})$ **o** $(-\sqrt{2})^3(3-\sqrt{2})$

Example 10 **Self Tutor**

Expand and simplify: $(3-\sqrt{2})(4+2\sqrt{2})$

$$\begin{aligned}&(3-\sqrt{2})(4+2\sqrt{2})\\&=12+3\times 2\sqrt{2}+(-\sqrt{2})\times 4+(-\sqrt{2})\times 2\sqrt{2} \quad \{\text{FOIL rule}\}\\&=12+6\sqrt{2}-4\sqrt{2}-4\\&=8+2\sqrt{2}\end{aligned}$$

3 Expand and simplify:

a $(1+\sqrt{2})(2+\sqrt{2})$ **b** $(2+\sqrt{3})(2+\sqrt{3})$ **c** $(\sqrt{3}+2)(\sqrt{3}-1)$

d $(4-\sqrt{2})(3+\sqrt{2})$ **e** $(1+\sqrt{3})(1-\sqrt{3})$ **f** $(5+\sqrt{7})(2-\sqrt{7})$

g $(\sqrt{5}+2)(\sqrt{5}-3)$ **h** $(2\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **i** $(4-\sqrt{2})(3-\sqrt{2})$

Example 11 **Self Tutor**

Expand and simplify:

a $(\sqrt{3}+2)^2$ **b** $(\sqrt{3}-\sqrt{7})^2$

a
$$\begin{aligned}&(\sqrt{3}+2)^2\\&=(\sqrt{3})^2 \;+\; 2\times\sqrt{3}\times 2 \;+\; 2^2\\&=3+4\sqrt{3}+4\\&=7+4\sqrt{3}\end{aligned}$$

b
$$\begin{aligned}&(\sqrt{3}-\sqrt{7})^2\\&=(\sqrt{3})^2 \;+\; 2\times\sqrt{3}\times(-\sqrt{7}) \;+\; (-\sqrt{7})^2\\&=3-2\sqrt{21}+7\\&=10-2\sqrt{21}\end{aligned}$$

4 Expand and simplify:

a $(1+\sqrt{2})^2$ **b** $(2-\sqrt{3})^2$ **c** $(\sqrt{3}+2)^2$

d $(1+\sqrt{5})^2$ **e** $(\sqrt{2}-\sqrt{3})^2$ **f** $(5-\sqrt{2})^2$

g $(\sqrt{2}+\sqrt{7})^2$ **h** $(4-\sqrt{6})^2$ **i** $(\sqrt{6}-\sqrt{2})^2$

j $(\sqrt{5}+2\sqrt{2})^2$ **k** $(\sqrt{5}-2\sqrt{2})^2$ **l** $(6+\sqrt{8})^2$

m $(5\sqrt{2}-1)^2$ **n** $(3-2\sqrt{2})^2$ **o** $(1+3\sqrt{2})^2$

5 Use the binomial expansion for $(a+b)^3$ to write the following in simplest radical form:

a $(3+\sqrt{7})^3$ **b** $(\sqrt{3}-\sqrt{2})^3$

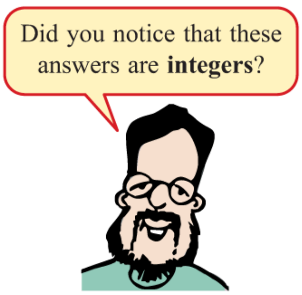
Example 12 **Self Tutor**

Expand and simplify:

a $(3+\sqrt{2})(3-\sqrt{2})$ **b** $(2\sqrt{3}-5)(2\sqrt{3}+5)$

a
$$\begin{aligned}&(3+\sqrt{2})(3-\sqrt{2})\\&=3^2-(\sqrt{2})^2\\&=9-2\\&=7\end{aligned}$$

b
$$\begin{aligned}&(2\sqrt{3}-5)(2\sqrt{3}+5)\\&=(2\sqrt{3})^2-5^2\\&=(4\times 3)-25\\&=12-25\\&=-13\end{aligned}$$



6 Expand and simplify:

a $(4+\sqrt{3})(4-\sqrt{3})$ **b** $(5-\sqrt{2})(5+\sqrt{2})$ **c** $(\sqrt{5}-2)(\sqrt{5}+2)$

d $(\sqrt{7}+4)(\sqrt{7}-4)$ **e** $(3\sqrt{2}+2)(3\sqrt{2}-2)$ **f** $(2\sqrt{5}-1)(2\sqrt{5}+1)$

g $(5-3\sqrt{3})(5+3\sqrt{3})$ **h** $(2-4\sqrt{2})(2+4\sqrt{2})$ **i** $(1+5\sqrt{7})(1-5\sqrt{7})$

7 Expand and simplify:

a $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$ **b** $(\sqrt{7}+\sqrt{11})(\sqrt{7}-\sqrt{11})$ **c** $(\sqrt{x}-\sqrt{y})(\sqrt{y}+\sqrt{x})$

d $(3\sqrt{2}+\sqrt{3})(3\sqrt{2}-\sqrt{3})$ **e** $(3\sqrt{3}+\sqrt{7})(3\sqrt{3}-\sqrt{7})$ **f** $(2\sqrt{5}-3\sqrt{2})(3\sqrt{2}+2\sqrt{5})$

E DIVISION BY RADICALS

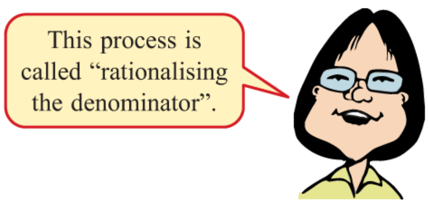
When an expression involves division by a radical, we can write the expression with an **integer denominator** which does **not** contain radicals.

If the denominator contains a simple radical such as $\sqrt{a}$ then we use the rule $\sqrt{a}\times\sqrt{a}=a$.

For example, consider $\dfrac{6}{\sqrt{3}}$.

Since $\dfrac{\sqrt{3}}{\sqrt{3}}=1$, we can multiply $\dfrac{6}{\sqrt{3}}$ by $\dfrac{\sqrt{3}}{\sqrt{3}}$ without changing its value.

$$\begin{aligned}\frac{6}{\sqrt{3}}&=\frac{6}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{6\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}\\&=\frac{6\sqrt{3}}{3}\qquad\{\text{since }\sqrt{a}\times\sqrt{a}=a\}\\&=2\sqrt{3}\end{aligned}$$



Example 13 **Self Tutor**

Express with integer denominator:

a $\dfrac{7}{\sqrt{3}}$ **b** $\dfrac{10}{\sqrt{5}}$ **c** $\dfrac{10}{2\sqrt{2}}$

a $$\begin{aligned}&\frac{7}{\sqrt{3}}\\&=\frac{7}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{7\sqrt{3}}{3}\end{aligned}$$

b $$\begin{aligned}&\frac{10}{\sqrt{5}}\\&=\frac{10}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}\\&=\frac{10\sqrt{5}}{5}\\&=2\sqrt{5}\end{aligned}$$

c $$\begin{aligned}&\frac{10}{2\sqrt{2}}\\&=\frac{10}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\&=\frac{10\sqrt{2}}{4}\\&=\frac{5\sqrt{2}}{2}\end{aligned}$$

Now suppose the denominator has the form $a+b\sqrt{c}$.

The **radical conjugate** of $a+b\sqrt{c}$ is $a-b\sqrt{c}$.

The product of the radical conjugates is $(a+b\sqrt{c})(a-b\sqrt{c}) = a^2-(b\sqrt{c})^2$
$= a^2 - b^2c$

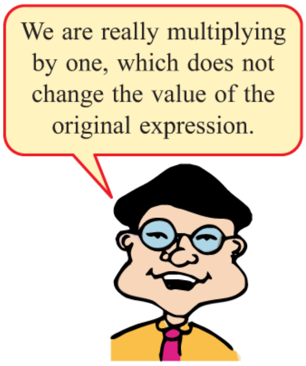
which does not involve a radical.

So, given a denominator of the form $a+b\sqrt{c}$, we multiply the fraction by $\dfrac{a-b\sqrt{c}}{a-b\sqrt{c}}$.

Example 14 **Self Tutor**

Express $\dfrac{1}{3+5\sqrt{2}}$ with integer denominator.

$$\begin{aligned}\frac{1}{3+5\sqrt{2}} &= \left(\frac{1}{3+5\sqrt{2}}\right)\left(\frac{3-5\sqrt{2}}{3-5\sqrt{2}}\right)\\ &= \frac{3-5\sqrt{2}}{3^2-(5\sqrt{2})^2} \qquad \{\text{using } (a+b)(a-b)=a^2-b^2\}\\ &= \frac{3-5\sqrt{2}}{9-50}\\ &= \frac{5\sqrt{2}-3}{41}\end{aligned}$$



EXERCISE 4E

1 Express with integer denominator:

a $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{2}{\sqrt{2}}$ **c** $\dfrac{4}{\sqrt{2}}$ **d** $\dfrac{\sqrt{3}}{\sqrt{2}}$ **e** $\dfrac{\sqrt{7}}{3\sqrt{2}}$

f $\dfrac{1}{\sqrt{3}}$ **g** $\dfrac{3}{\sqrt{3}}$ **h** $\dfrac{4}{\sqrt{3}}$ **i** $\dfrac{\sqrt{7}}{\sqrt{3}}$ **j** $\dfrac{\sqrt{11}}{4\sqrt{3}}$

k $\dfrac{1}{\sqrt{5}}$ **l** $\dfrac{3}{\sqrt{5}}$ **m** $\dfrac{15}{\sqrt{5}}$ **n** $\dfrac{\sqrt{3}}{\sqrt{5}}$ **o** $\dfrac{125}{2\sqrt{5}}$

p $\dfrac{\sqrt{10}}{\sqrt{2}}$ **q** $\dfrac{1}{2\sqrt{3}}$ **r** $\dfrac{2\sqrt{2}}{\sqrt{3}}$ **s** $\dfrac{15}{2\sqrt{5}}$ **t** $\dfrac{1}{(\sqrt{2})^3}$

2 Express with integer denominator:

a $\dfrac{1}{3-\sqrt{5}}$ **b** $\dfrac{1}{2+\sqrt{3}}$ **c** $\dfrac{1}{4-\sqrt{11}}$ **d** $\dfrac{\sqrt{2}}{5+\sqrt{2}}$

e $\dfrac{\sqrt{3}}{3+\sqrt{3}}$ **f** $\dfrac{5}{2-3\sqrt{2}}$ **g** $\dfrac{-\sqrt{5}}{3+2\sqrt{5}}$ **h** $\dfrac{3-2\sqrt{7}}{2+3\sqrt{7}}$

3 Write in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$:

a $\dfrac{4}{2-\sqrt{2}}$ **b** $\dfrac{-5}{1+\sqrt{2}}$ **c** $\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$ **d** $\dfrac{\sqrt{2}-2}{3-\sqrt{2}}$

e $\dfrac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$ **f** $\dfrac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$ **g** $\dfrac{1}{1-\frac{\sqrt{2}}{3}}$ **h** $\dfrac{\frac{\sqrt{2}}{2}+1}{1-\frac{\sqrt{2}}{4}}$

Example 15 **Self Tutor**

Write $\dfrac{\sqrt{3}}{1-\sqrt{3}} - \dfrac{1-2\sqrt{3}}{1+\sqrt{3}}$ in simplest form.

$$\begin{aligned}
\frac{\sqrt{3}}{1-\sqrt{3}} - \frac{1-2\sqrt{3}}{1+\sqrt{3}} &= \left(\frac{\sqrt{3}}{1-\sqrt{3}}\right)\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right) - \left(\frac{1-2\sqrt{3}}{1+\sqrt{3}}\right)\left(\frac{1-\sqrt{3}}{1-\sqrt{3}}\right)\\
&= \frac{\sqrt{3}+3}{1-3} - \frac{1-\sqrt{3}-2\sqrt{3}+6}{1-3}\\
&= \frac{\sqrt{3}+3}{-2} - \frac{7-3\sqrt{3}}{-2}\\
&= \frac{\sqrt{3}+3-7+3\sqrt{3}}{-2}\\
&= \frac{-4+4\sqrt{3}}{-2}\\
&= \frac{-4}{-2} + \frac{4\sqrt{3}}{-2}\\
&= 2-2\sqrt{3}
\end{aligned}$$

4 Write in simplest form:

a $\dfrac{1+\sqrt{2}}{1-\sqrt{2}} + \dfrac{1-\sqrt{2}}{1+\sqrt{2}}$ **b** $\dfrac{2+\sqrt{5}}{2-\sqrt{5}} - \dfrac{\sqrt{5}}{2+\sqrt{5}}$ **c** $\dfrac{4-\sqrt{3}}{3-2\sqrt{2}} - \dfrac{2\sqrt{3}}{3+2\sqrt{2}}$

5 **a** Suppose a and b are positive integers. Show that $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ is also an integer.

b Write with an integer denominator:

i $\dfrac{4}{\sqrt{7}+\sqrt{2}}$ **ii** $\dfrac{2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ **iii** $\dfrac{\sqrt{3}+2\sqrt{13}}{\sqrt{3}-\sqrt{13}}$

6 Write $\sqrt{\dfrac{3+2\sqrt{2}}{3-2\sqrt{2}}}$ in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

7 If $\dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{2}} = p$, find $\sqrt{6}$ in terms of p.

8 If $x = \sqrt{5}-\sqrt{3}$, find x^2 and hence show that $x^4 - 16x^2 + 4 = 0$.
You have just shown that one solution of $x^4 - 16x^2 + 4 = 0$ is $x = \sqrt{5}-\sqrt{3}$.

9 Suppose $u_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$. Evaluate u_n for $n = 1, 2, 3, 4, 5$, and 6.

F EQUALITY OF SURDS

We have discussed how irrational radicals such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$ are also known as **surds**. In this section we develop a theorem for the equality of surds which does *not* hold for rational radicals.

Following is the traditional **proof by contradiction** that $\sqrt{2}$ is an irrational number.

Proof:

Suppose $\sqrt{2}$ is rational.

$\therefore$ $\sqrt{2} = \frac{p}{q}$ where p and q are integers, $q \neq 0$, and where all common factors of p and q have been cancelled.

Now $2 = \frac{p^2}{q^2}$ {squaring both sides}

$\therefore$ $p^2 = 2q^2$

$\therefore$ p^2 is even {as it has a factor of 2 and q^2 is an integer}

$\therefore$ p is even. .... (1)

$\therefore$ we can write $p = 2k$ where k is some integer

$\therefore$ $(2k)^2 = 2q^2$

$\therefore$ $2q^2 = 4k^2$

$\therefore$ $q^2 = 2k^2$ where k^2 is an integer.

$\therefore$ q^2 is even and so q is even .... (2)

From (1) and (2) we have a contradiction to our supposition, since if both p and q are even then they share a common factor of 2.

$\therefore$ the supposition is false, and hence $\sqrt{2}$ must be irrational.

An immediate consequence of $\sqrt{2}$ being irrational is:

If a, b, c, and d are rational, and $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

Proof:

Suppose a, b, c, and d are rational. Assume that $b \neq d$.

So, $a + b\sqrt{2} = c + d\sqrt{2}$ gives

$a - c = (d - b)\sqrt{2}$ .... (1)

$\therefore$ $\frac{a - c}{d - b} = \sqrt{2}$ where the LHS exists as $b \neq d$.

However, this result is impossible as the LHS is rational and the RHS is irrational.

Thus the assumption is false, and so $b = d$.

$\therefore$ using (1), $a - c = 0$, and so $a = c$.

THEOREM FOR EQUALITY OF SURDS

Suppose $\sqrt{k}$ is irrational, and that a, b, c, and d are rational.
If $a + b\sqrt{k} = c + d\sqrt{k}$ then $a = c$ and $b = d$.

We can easily show by counter-example that this theorem does not hold for rational radicals.

For example, $1 + 4\sqrt{4} = 3 + 3\sqrt{4}$ but $1 \neq 3$ and $4 \neq 3$.

Example 16 — Self Tutor

Solve for x and y given that they are rational:

a $x + y\sqrt{2} = 5 - 6\sqrt{2}$

b $(x + y\sqrt{2})(3 - \sqrt{2}) = -2\sqrt{2}$

a Since $\sqrt{2}$ is irrational, $x = 5$ and $y = -6$.

b
$$(x + y\sqrt{2})(3 - \sqrt{2}) = -2\sqrt{2}$$
$$\therefore\ x + y\sqrt{2} = \frac{-2\sqrt{2}}{3 - \sqrt{2}}$$
$$= \left(\frac{-2\sqrt{2}}{3 - \sqrt{2}}\right)\left(\frac{3 + \sqrt{2}}{3 + \sqrt{2}}\right)$$
$$= \frac{-6\sqrt{2} - 4}{9 - 2}$$
$$= \frac{-4 - 6\sqrt{2}}{7}$$
$$= -\tfrac{4}{7} - \tfrac{6}{7}\sqrt{2}$$
$\therefore$ since $\sqrt{2}$ is irrational, $x = -\frac{4}{7}$ and $y = -\frac{6}{7}$.

Example 17 — Self Tutor

Find rationals a and b such that $(a + 2\sqrt{2})(3 - \sqrt{2}) = 5 + b\sqrt{2}$.

$$(a + 2\sqrt{2})(3 - \sqrt{2}) = 5 + b\sqrt{2}$$
$$\therefore\ 3a - a\sqrt{2} + 6\sqrt{2} - 4 = 5 + b\sqrt{2}$$
$$\therefore\ (3a - 4) + (6 - a)\sqrt{2} = 5 + b\sqrt{2}$$
$\therefore$ since $\sqrt{2}$ is irrational, $3a - 4 = 5$ and $6 - a = b$
$$\therefore\ 3a = 9$$
$$\therefore\ a = 3 \quad \text{and hence} \quad b = 6 - 3 = 3$$

EXERCISE 4F

1 Solve for x and y given that they are rational:

a $x + y\sqrt{2} = 3 + 2\sqrt{2}$

b $15 - 4\sqrt{2} = x + y\sqrt{2}$

c $-x + y\sqrt{2} = 11 - 3\sqrt{2}$

d $x + y\sqrt{2} = 6$

e $x + y\sqrt{2} = -3\sqrt{2}$

f $x + y\sqrt{2} = 0$

2 Solve for x and y given that they are rational:

a $(x + y\sqrt{2})(2 - \sqrt{2}) = 1 + \sqrt{2}$

b $(x + y\sqrt{2})(3 + \sqrt{2}) = 1$

c $(2 - 3\sqrt{2})(x + y\sqrt{2}) = \sqrt{2}$

d $(x + y\sqrt{2})(3 - \sqrt{2}) = -4\sqrt{2}$

3 Find rationals a and b such that:

a $(a + \sqrt{2})(2 - \sqrt{2}) = 4 - b\sqrt{2}$

b $(a + 3\sqrt{2})(3 - \sqrt{2}) = 6 + b\sqrt{2}$

c $(a + b\sqrt{2})^2 = 33 + 20\sqrt{2}$

d $(a + b\sqrt{2})^2 = 41 - 24\sqrt{2}$

4 Find $\sqrt{11 - 6\sqrt{2}}$. **Hint:** $\sqrt{2}$ is never negative.

5 **a** Write $\sqrt{11+4\sqrt{6}}$ in the form $a\sqrt{2}+b\sqrt{3}$ where $a, b \in \mathbb{Q}$.

b Can $\sqrt{11+4\sqrt{6}}$ be written in the form $a+b\sqrt{6}$ where $a, b \in \mathbb{Q}$? Explain your answer.

INVESTIGATION — CONTINUED SQUARE ROOTS

$X = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+....}}}}}$ is an example of a **continued square root**.

Some continued square roots can be simplified to integers.

What to do:

1 Use your calculator to find, correct to 6 decimal places:

a $\sqrt{2}$ **b** $\sqrt{2+\sqrt{2}}$ **c** $\sqrt{2+\sqrt{2+\sqrt{2}}}$

d $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$ **e** $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}$

2 Continue the process and hence predict the value of X.

3 Use algebra to find the value of X.
Hint: Find X^2 in terms of X.

4 Work your algebraic solution in **3** backwards to find a continued square root whose value is 3.

REVIEW SET 4A

1 Simplify:

a $(3\sqrt{2})^2$ **b** $-2\sqrt{3}\times 4\sqrt{3}$ **c** $3\sqrt{2}-\sqrt{8}$

d $(\sqrt[3]{-27})^3$ **e** $(5\sqrt{3})^3 \times (\sqrt{2})^4$ **f** $\sqrt{\frac{81}{256}}$

2 Write in simplest radical form:

a $\sqrt{48}$ **b** $\sqrt{864}$

3 Simplify:

a $2\sqrt{3}+6\sqrt{5}-3\sqrt{3}-4\sqrt{5}$ **b** $\frac{\sqrt{6}}{3}-\frac{\sqrt{6}}{4}+\frac{2\sqrt{6}}{5}$

4 Expand and simplify:

a $2\sqrt{3}(4-\sqrt{3})$ **b** $(3-\sqrt{7})^2$ **c** $(2-\sqrt{3})(2+\sqrt{3})$

d $(3+2\sqrt{5})(2-\sqrt{5})$ **e** $(4-\sqrt{2})(3+2\sqrt{2})$ **f** $(\sqrt{7}+3\sqrt{8})^2$

5 Express with integer denominator:

a $\frac{8}{\sqrt{2}}$ **b** $\frac{15}{\sqrt{3}}$ **c** $\frac{\sqrt{3}}{4+\sqrt{2}}$ **d** $\frac{5}{6-2\sqrt{3}}$

6 Write $\sqrt{\frac{1}{7}}$ in the form $k\sqrt{7}$.

7 Use the binomial expansion for $(a+b)^3$ to write $(5+\sqrt{2})^3$ in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

8 Write in the form $a+b\sqrt{3}$ where $a, b \in \mathbb{Q}$:

a $\dfrac{\frac{\sqrt{3}}{2}+1}{1-\frac{\sqrt{3}}{2}}$ **b** $\dfrac{2+\sqrt{3}}{2-\sqrt{3}} - \dfrac{2\sqrt{3}}{2+\sqrt{3}}$

9 Write with integer denominator: $\dfrac{2\sqrt{5}-\sqrt{7}}{\sqrt{5}+2\sqrt{7}}$

10 Find $x, y \in \mathbb{Q}$ such that $(3+x\sqrt{5})(\sqrt{5}-y) = -13+5\sqrt{5}$.

REVIEW SET 4B

1 Simplify:

a $2\sqrt{3}\times 3\sqrt{5}$ **b** $(2\sqrt{5})^2$ **c** $5\sqrt{2}-7\sqrt{2}$

d $-\sqrt{2}(2-\sqrt{2})$ **e** $(\sqrt{3})^4$ **f** $\sqrt{3}\times\sqrt{5}\times\sqrt{15}$

2 Write $\sqrt{75}$ in simplest radical form.

3 Write in simplest radical form $a+b\sqrt{n}$ where $a, b \in \mathbb{Q}$, $n \in \mathbb{Z}$:

a $\dfrac{3+\sqrt{24}}{2}$ **b** $\dfrac{8-\sqrt{72}}{4}$

4 Expand and simplify:

a $(5-\sqrt{3})(5+\sqrt{3})$ **b** $-(2-\sqrt{5})^2$

c $2\sqrt{3}(\sqrt{3}-1)-2\sqrt{3}$ **d** $(2\sqrt{2}-5)(1-\sqrt{2})$

5 Express with integer denominator:

a $\dfrac{14}{\sqrt{2}}$ **b** $\dfrac{\sqrt{2}}{1-\sqrt{3}}$ **c** $\dfrac{\sqrt{2}}{1+3\sqrt{2}}$ **d** $\dfrac{-5}{5-2\sqrt{3}}$

6 Use the binomial expansion $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ to write $(\sqrt{2}+\sqrt{3})^4$ in simplest radical form.

7 Write in the form $a+b\sqrt{5}$ where $a, b \in \mathbb{Q}$:

a $\dfrac{1-\frac{1}{\sqrt{5}}}{2\sqrt{5}+\frac{1}{\sqrt{5}}}$ **b** $\dfrac{3-\sqrt{5}}{3+\sqrt{5}} - \dfrac{4}{3-\sqrt{5}}$ **c** $\sqrt{\dfrac{3-\sqrt{5}}{3+\sqrt{5}}}$

8 Write in simplest form:

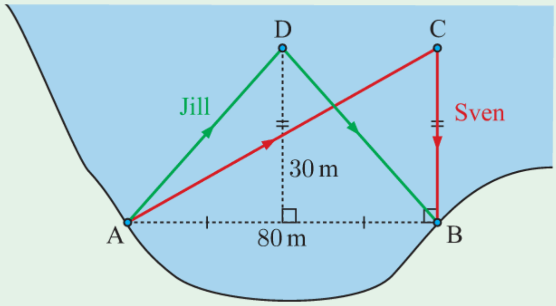
a $\dfrac{2-\sqrt{2}}{1+\sqrt{2}} - \dfrac{1+\sqrt{2}}{1-\sqrt{2}}$ **b** $\dfrac{2+\sqrt{3}}{1+2\sqrt{3}} + \dfrac{-2\sqrt{3}}{1-2\sqrt{3}}$

9 Write $\sqrt{\dfrac{3\sqrt{2}+1}{3\sqrt{2}-1}}$ in the form $a\sqrt{b}+c\sqrt{d}$.

10 Find $p, q \in \mathbb{Q}$ such that $(p+3\sqrt{7})(5+q\sqrt{7}) = 9\sqrt{7}-53$.

OPENING PROBLEM

Sven has challenged Jill to a swimming race in a lake. Both swimmers will swim from A to B, but each needs to touch a buoy along the way. Sven thinks he is a very good swimmer, so he will swim to buoy C on his way to B. Jill will swim to buoy D, as shown.



Things to think about:

a How far will Sven swim?

b How far will Jill swim?

c How much further will Sven swim than Jill?

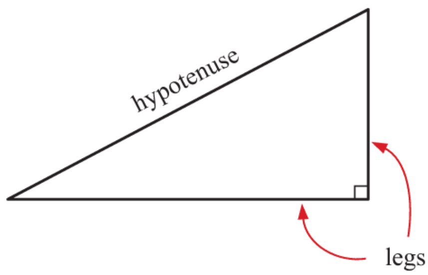
A PYTHAGORAS' THEOREM

A **right angled triangle** is a triangle which has a right angle as one of its angles.

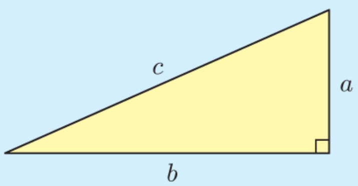
The side opposite the right angle is called the **hypotenuse**, and is the **longest** side of the triangle.

The other two sides are called the **legs** of the triangle.

Around 500 BC, the Greek mathematician **Pythagoras** discovered a rule which connects the lengths of the sides of right angled triangles.



PYTHAGORAS' THEOREM



In a right angled triangle with hypotenuse c and legs a and b, $c^2 = a^2 + b^2$.

GEOMETRY PACKAGE

RESEARCH

The Persian mathematician **Al-Nayrīzī** (865 - 922 AD) came from Nayriz, Iran. Around 900 AD, he used the tiling arrangement shown to prove Pythagoras' theorem.

Research the method of **five-piece dissection** he used for his proof.