INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 5.20-5.21]

COUPLED DIFFERENTIAL EQUATIONS

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A. Paper 1 questions (SHORT)

SIMPLE SYSTEMS OF COUPLED DIFFERENTIAL EQUATIONS

1. [Maximum mark: 7]

(a) Solve the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 3x$, $t = 0$, $x = 5$. Solution in the form $x = A\mathrm{e}^{bt}$ [3]

(b) Solve the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 4y \qquad \text{with} \qquad t = 0,\ (x, y) = (5, 6)$$ [4]

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2. [Maximum mark: 7]

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 8x$$

(a) Find the general solution. [5]

(b) Find the particular solution given that $(x, y) = (5, 2)$ when $t = 0$. [2]

3. [Maximum mark: 7]

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - y\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$$

(a) Find the general solution. [5]

(b) Find the particular solution given that $(x, y) = (2, 3)$ when $t = 0$. [2]

4. [Maximum mark: 7]

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 2y\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - y$$

(a) Find the general solution. [5]

(b) Find the particular solution given that $(x, y) = (3, 2)$ when $t = 0$. [2]

5. [Maximum mark: 7]

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 4y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 2y$$

(a) Find the general solution. [5]

(b) Find the particular solution given that $(x, y) = (9, 1)$ when $t = 0$. [2]

2nd ORDER DIFFERENTIAL EQUATIONS

6. [Maximum mark: 7]

Solve the 2nd order differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 5\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0$$

with $x = 3$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 6$, when $t = 0$.

7. [Maximum mark: 7]

Solve the 2nd order differential equation

$$\ddot{x} - 9x = 0$$

with $x = 4$, $\dot{x} = 0$, when $t = 0$.

EULER'S METHOD FOR COUPLED DIFFERENTIAL EQUATIONS

8. [Maximum mark: 9]

Consider the system of coupled differential equations [**of question 3**]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$$

given that $(x, y) = (2, 3)$ when $t = 0$.

The particular solution is

$$x = \mathrm{e}^{2t} + \mathrm{e}^{3t}$$
$$y = 2\mathrm{e}^{2t} + \mathrm{e}^{3t}$$

(a) Find the values of x and y when $t = 1$, **correct to 3 dp**. [2]

(b) By using Euler's method with step $h = 0.2$,

(i) show that the first step gives $x_1 = 3$, $y_1 = 4.4$

(ii) find approximate values for x and y when $t = 1.$, **correct to 3 dp.** [5]

(c) Find a better approximation when $t = 1$, **correct to 3 dp**, by using Euler's method with step $h = 0.01$. [2]

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9. [Maximum mark: 7]

Consider the system of coupled differential equations [**of question 4**]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 2y\,,\ \frac{\mathrm{d}y}{\mathrm{d}t} = -x - y\,,$$

given that $(x, y) = (3, 2)$ when $t = 0$.

The particular solution is

$$x = \mathrm{e}^{-2t} + 2\mathrm{e}^{-3t}$$
$$y = \mathrm{e}^{-2t} + \mathrm{e}^{-3t}$$

(a) Find the values of x and y when $t = 1$, **correct to 5 dp**. [2]

(b) By using Euler's method, find approximate values for x and y when $t = 1$, **correct to 5 dp**.

(i) when $h = 0.2$

(ii) when $h = 0.01$. [5]

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10. [Maximum mark: 7]

Consider the system of coupled differential equations [**of question 5**]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 4y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 2y,$$

given that $(x, y) = (9,1)$ when $t = 0$.

The particular solution is

$$x = 8\mathrm{e}^{2t} + \mathrm{e}^{-3t}$$
$$y = 2\mathrm{e}^{2t} - \mathrm{e}^{-3t}$$

(a) Find the values of x and y when $t = 1$, correct to **3 dp**. [2]

(b) By using Euler's method, find approximate values for x and y when $t = 1$, correct to **3 dp**

(i) when $h = 0.2$

(ii) when $h = 0.01$. [5]

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11. [Maximum mark: 5]

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 4y + 1, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - 2y + 3t + 2,$$

given that $(x, y) = (2, 3)$ when $t = 0$.

By using Euler's method with step $h = 0.1$, find approximate values for x and y when $t = 1$, correct to **3 dp**

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12. [Maximum mark: 5]

Consider the system of coupled differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy + t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x + y - t + 1,$$

given that $(x, y) = (2, 3)$ when $t = 1$.

By using Euler's method with step $h = 0.1$, find approximate values for x and y when $t = 2$, correct to **3 sf.**

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PHASE PORTRAIT

13. [Maximum mark: 6]

Consider the system of coupled differential equations [**of question 3**]

$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x - y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 2x + y$ with general solution $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + B\mathrm{e}^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(a) Sketch the phase portrait of the general solution. [3]

(b) Sketch the curve of the particular solution given that $(x, y) = (2, 3)$ when $t = 0$. [2]

(c) Describe the equilibrium point (0.0). [1]

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14. [Maximum mark: 6]

Consider the system of coupled differential equations [**of question 4**]

$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 2y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -x - y$ with general solution $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(a) Sketch the phase portrait of the general solution. [3]

(b) Sketch the curve of the particular solution given that $(x, y) = (3, 2)$ when $t = 0$. [2]

(c) Describe the equilibrium point (0.0). [1]

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15. [Maximum mark: 6]

Consider the system of coupled differential equations [**of question 5**]

$\frac{\mathrm{d}x}{\mathrm{d}t} = x + 4y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = x - 2y$ with general solution $\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{2t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + B\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(a) Sketch the phase portrait of the general solution. [3]

(b) Sketch the curve of the particular solution given that $(x, y) = (9,1)$ when $t = 0$. [2]

(c) Describe the equilibrium point (0.0). [1]

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16. [Maximum mark: 6]

Find the equilibrium point for each of the following systems of coupled differential equations

(a) $\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 2y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = -x - y$ [2]

(b) $\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 2y$, $\frac{\mathrm{d}y}{\mathrm{d}t} = x + y - 3$ [3]

(c) $\frac{\mathrm{d}x}{\mathrm{d}t} = 3x + 2y - 12$, $\frac{\mathrm{d}y}{\mathrm{d}t} = x + 5y - 17$ [3]

17. [Maximum mark: 10]

The line $y = x$ is parallel to the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The line $y = -x$ is parallel to the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The first column of the following table shows details of the general solution of a system of coupled differential equations

Column 1 general solution	**Column 2** (A to F)	**Column 3** equilibrium point (0,0)
$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{7t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$		
$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-7t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$		
$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathrm{e}^{-7t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + B\mathrm{e}^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$		
Eigenvalues $\lambda = \pm 3\mathrm{i}$		
Eigenvalues $\lambda = -2 \pm 3\mathrm{i}$		
Eigenvalues $\lambda = 2 \pm 3\mathrm{i}$		

Complete **Column 2** by using one of the letters A to F below, which correspond to a description of the trajectories of the system.

A. All trajectories away from (0,0). Curved trajectories tend to be parallel to $y = x$

B. All trajectories away from (0,0). Curved trajectories tend to be parallel to $y = -x$

C. All trajectories towards (0,0). Curved trajectories start almost parallel to $y = x$

D. All trajectories towards (0,0). Curved trajectories start almost parallel to $y = -x$

E. Trajectories towards (0,0) parallel to $y = x$, then away from (0,0) parallel to $y = -x$.

F. Trajectories towards (0,0) parallel to $y = -x$, then away from (0,0) parallel to $y = x$.

G. Spiral trajectories clockwise away from (0,0)

H. Spiral trajectories clockwise towards from (0,0)

I. Ellipse trajectories

Complete **Column 3** with one of the terms below that describe the equilibrium point.

saddle **stable** **spiral-stable**

center **unstable** **spiral-unstable**