

EXERCISES [MAI 5.2-5.3]
BASIC DERIVATIVES – TANGENT AND NORMAL
SOLUTIONS

A. Paper 1 questions (SHORT)

1.

Function	Derivative
$y = 7x^3 + 5x^2 + 2x + 3$	$\frac{dy}{dx} = 21x^2 + 10x + 2$
$y = \frac{7}{3}x^3 - \frac{5}{2}x^2 + \frac{1}{3}x + \frac{4}{5}$	$\frac{dy}{dx} = 7x^2 - 5x + \frac{1}{3}$
$y = \frac{7x^3}{3} - \frac{5x^2}{2} + \frac{x}{3} + \frac{4}{5}$	$\frac{dy}{dx} = 7x^2 - 5x + \frac{1}{3}$ (it's the same)
$y = 1 + \frac{2}{x} + \frac{3}{x^2}$ ($= 1 + 2x^{-1} + 3x^{-2}$)	$\frac{dy}{dx} = -\frac{2}{x^2} - \frac{6}{x^3}$
$y = \frac{1}{3} + \frac{2}{5x} + \frac{3}{7x^2}$ ($= \frac{1}{3} + \frac{2}{5}x^{-1} + \frac{3}{7}x^{-2}$)	$\frac{dy}{dx} = -\frac{2}{5x^2} - \frac{6}{7x^3}$
$y = x^2(1 + \frac{2}{x} + \frac{3}{x^2})$ ($= x^2 + 2x + 3$)	$\frac{dy}{dx} = 2x + 2$
$y = \sqrt{x} + \sqrt[3]{x}$ ($= x^{\frac{1}{2}} + x^{\frac{1}{3}}$)	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{3}x^{-\frac{2}{3}}$ ($= \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}$)
$y = \sqrt{x^3} + \sqrt[3]{x^2}$ ($= x^{\frac{3}{2}} + x^{\frac{2}{3}}$)	$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} + \frac{2}{3}\sqrt[3]{x^{-1}}$ ($= \frac{3}{2}\sqrt{x} + \frac{2}{3\sqrt[3]{x}}$)
$y = \frac{1+x+x^2}{x^2}$ ($= x^{-2} + x^{-1} + 1$)	$\frac{dy}{dx} = -2x^{-3} - x^{-2}$ ($= -\frac{2}{x^3} - \frac{1}{x^2}$)
$y = \frac{3+5x+7x^2}{2x^2}$ ($= \frac{3}{2}x^{-2} + \frac{5}{2}x^{-1} + \frac{7}{2}$)	$\frac{dy}{dx} = -3x^{-3} - \frac{5}{2}x^{-2}$ ($= -\frac{3}{x^3} - \frac{5}{2x^2}$)

2. (a) $f'(x) = 10x$

(b) $f'(1) = 10$

(c) $f'(x) = 20 \Leftrightarrow 10x = 20 \Leftrightarrow x = 2$, $y = 23$, thus (5,23)

3. (a) $f'(x) = \frac{2}{\sqrt{x}}$

(b) $f'(1) = 2$

(c) $f'(x) = 1 \Leftrightarrow \frac{2}{\sqrt{x}} = 1 \Leftrightarrow \sqrt{x} = 2 \Leftrightarrow x = 4$, $y = 8$, thus (4,8)

4. (a) $f'(x) = 3x^2 - 4x - 0 = 3x^2 - 4x$
 (b) Gradient $= f'(2) = 3 \times 4 - 4 \times 2 = 4$
5. $f'(x) = 2x - 3b$
 $f(1) = 0 \Leftrightarrow 1^2 - 3b + c + 2 = 0 \Leftrightarrow -3b + c + 3 = 0$
 $f'(3) = 0 \Leftrightarrow 6 - 3b = 0 \Rightarrow 3b = 6 \Rightarrow b = 2$
 $-6 + c + 3 = 0 \Rightarrow c = 3$
6. (a) At $x = 1$
 (i) $y = f(1) + g(1) = 2 + 3 = 5$
 (ii) $y = 2f(1) + 3g(1) = 4 + 9 = 13$
 (iii) $y = f(1) + 5 = 2 + 5 = 7$
- (b) (i) $\frac{dy}{dx} = f'(x) + g'(x)$. At $x = 1$, $\frac{dy}{dx} = f'(1) + g'(1) = 4 + 5 = 9$
 (ii) $\frac{dy}{dx} = 2f'(x) + 3g'(x)$. At $x = 1$, $\frac{dy}{dx} = 2f'(1) + 3g'(1) = 8 + 15 = 23$
 (iii) $\frac{dy}{dx} = f'(x) + 10x$. At $x = 1$, $\frac{dy}{dx} = f'(1) + 10 = 4 + 10 = 14$
7. (a) (i) 1 (ii) 0.5
 (b) (i) 0 (ii) $-\frac{1}{2}$
8. (a) (i) 1
 (ii) 2
 (iii) $f'(14) = f'(2)$ (or $f'(5)$ or $f'(8)$) $= -1$
- (b) There are five repeated periods of the graph, each with two solutions,
 ie number of solutions is $5 \times 2 = 10$

TANGENT LINE – NORMAL LINE

9. $f(x) = 2x^2 - 12x + 10$
 (a) $f'(x) = 4x - 12$
 (b) At $x = 2$, $y = 6$, Point (2,6)
 $m_T = -4$, $m_N = \frac{1}{4}$
 Tangent line: $y + 6 = -4(x - 2)$ (i.e. $y = -4x + 2$)
 Normal line: $y + 6 = \frac{1}{4}(x - 2)$ (i.e. $y = \frac{1}{4}x - \frac{13}{2}$)
- (a) At $x = 3$, $y = -8$, Point (3, -8)
 $m_T = 0$
 Tangent line: $y = -8$, Normal line: $x = 3$

10. $f(x) = 2x^2 - 12x + 10$

(a) $f'(x) = 4x - 12$

(b) (i) $m_1 = 4$

(ii) $4x - 12 = 4 \Leftrightarrow x = 4$, thus $y = -6$. Point (4, -6)

(c) (i) $m_2 = 8$

(ii) $4x - 12 = 8 \Leftrightarrow x = 5$, thus $y = 0$. Point (5, 0)

(iii) $y - 0 = 8(x - 5) \Leftrightarrow y = 8x - 40$

11. $y = x^3 + 1 \quad \frac{dy}{dx} = 3x^2$

At $x = 1$, $m_T = 3$, $m_N = -\frac{1}{3}$

Equation of tangent: $y - 2 = 3(x - 1) \Rightarrow y = 3x - 1$

Equation of normal: $y - 2 = -\frac{1}{3}(x - 1)$ (OR $x + 3y - 7 = 0$ OR $y = -\frac{1}{3}x + 2\frac{1}{3}$)

12. $f'(x) = 12x^2 + 2$

When $x = 1$, $f(1) = 6$, Point (1, 6)

When $x = 1$, $f'(1) = 14$, $m_T = 14$

Equation is $y - 6 = -\frac{1}{14}(x - 1)$ $\left(y = -\frac{1}{14}x + \frac{85}{14}, y = -0.0714x + 6.07 \right)$

13. $y = x^2 - x \quad \frac{dy}{dx} = 2x - 1$

Line parallel to $y = 5x \Rightarrow 2x - 1 = 5 \Rightarrow x = 3$ so $y = 6$, Point (3, 6)

14. $f'(x) = 4kx^3$

$m_N = -\frac{1}{8}$, thus $m_T = 8$

$4kx^3 = 8 \Rightarrow kx^3 = 2$

substituting $x = 1$, $k = 2$

15. (a) $f'(x) = 6x - 5$

(b) $f'(p) = 7 \Rightarrow 6p - 5 = 7 \Rightarrow p = 2$

(c) Setting $y(2) = f(2)$

$k + 2 = 5 \Rightarrow k = 3$

16. (a) $f(1) = 3 \Rightarrow p + q = 3$

$f'(x) = 2px + q$

$f'(1) = 8 \Rightarrow 2p + q = 8$

$p = 5, q = -2$

17. (a) **METHOD 1 (directly by GDC)**

The equation of the tangent is $y = -4x - 8$.

METHOD 2

For $x = -1$, $y = -4$ and $\frac{dy}{dx} = 3x^2 + 8x + 1$, $m_T = -4$

Therefore, the tangent equation is $y + 4 = -4(x + 1) \Rightarrow y = -4x - 8$.

(b) This tangent meets the curve when $-4x - 8 = x^3 + 4x^2 + x - 6$

The required point of intersection is $(-2, 0)$.

18. For the curve, $y = 7$ when $x = 1 \Rightarrow a + b = 14$, and

$$\frac{dy}{dx} = 6x^2 + 2ax + b = 16 \text{ when } x = 1 \Rightarrow 2a + b = 10.$$

Solving gives $a = -4$ and $b = 18$.

B. Paper 2 questions (LONG)

19. (a) (i) $p = 1, q = 5$ (or $p = 5, q = 1$)
(ii) $x = 3$ (must be an equation)

(b) $y = (x - 1)(x - 5) = x^2 - 6x + 5 = (x - 3)^2 - 4$ ($h = 3, k = -4$)

(c) $\frac{dy}{dx} = 2(x - 3)$ ($= 2x - 6$)

(d) When $x = 0$, $\frac{dy}{dx} = -6$
 $y - 5 = -6(x - 0)$ ($y = -6x + 5$ or equivalent)

20. (a) $h = 3$ $k = 2$

(b) $f(x) = -(x - 3)^2 + 2 = -x^2 + 6x - 9 + 2 = -x^2 + 6x - 7$

(c) $f'(x) = -2x + 6$

(d) (i) tangent gradient $= -2$ gradient of $L = \frac{1}{2}$

$$y - 1 = \frac{1}{2}(x - 4) \Rightarrow y = \frac{1}{2}x - 1$$

(ii) $-x^2 + 6x - 7 = \frac{1}{2}x - 1 \Leftrightarrow 2x^2 - 11x + 12 = 0 \Leftrightarrow x = 1.5 \text{ or } x = 4 \text{ so } x = 1.5$

(OR by GDC $-x^2 + 6x - 7 = \frac{1}{2}x - 1 \Rightarrow x = 1.5$)

21. (a) (i) $f'(x) = -x + 2$

(ii) $f'(0) = 2$

(b) Gradient of tangent at y -intercept $= f'(0) = 2$

$\Rightarrow$ gradient of normal $= \frac{1}{2}$ ($= -0.5$)

Therefore, equation of the normal is $y - 2.5 = -0.5(x - 0) \Rightarrow y = -0.5x + 2.5$

(c) (i) $-0.5x^2 + 2x + 2.5 = -0.5x + 2.5 \Rightarrow x = 0 \text{ or } x = 5$

(ii) Curve and normal intersect when $x = 0$ or $x = 5$

Other point is when $x = 5 \Rightarrow y = -0.5(5) + 2.5 = 0$ (so other point $(5, 0)$)

22. (a) (i) $p = -2$ $q = 4$ (or $p = 4, q = -2$)
- (ii) $y = a(x+2)(x-4) \Leftrightarrow 8 = a(6+2)(6-4) \Leftrightarrow 8 = 16a \Leftrightarrow a = \frac{1}{2}$
- (iii) $y = \frac{1}{2}(x+2)(x-4) \Rightarrow y = \frac{1}{2}(x^2 - 2x - 8) \Rightarrow y = \frac{1}{2}x^2 - x - 4$
- (b) $\frac{dy}{dx} = x - 1$
 $x - 1 = 7 \Leftrightarrow x = 8, y = 20$ (P is (8, 20))
- (c) (i) when $x = 4, m_T = 4 - 1 = 3 \Rightarrow m_N = -\frac{1}{3}$
 $y - 0 = -\frac{1}{3}(x - 4) \quad \left(y = -\frac{1}{3}x + \frac{4}{3} \right)$
- (ii) $\frac{1}{2}x^2 - x - 4 = -\frac{1}{3}x + \frac{4}{3} \Leftrightarrow x = -\frac{8}{3}$ or $x = 4$
 $x = -\frac{8}{3} \quad (-2.67)$

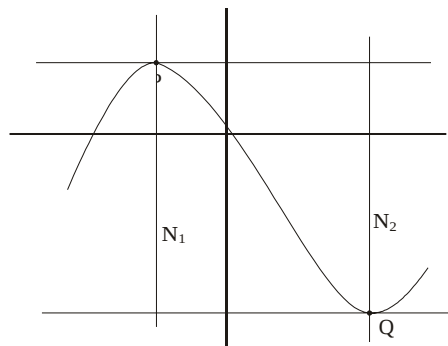
23. (a) $f'(x) = 3x^2 - 6x - 24$
- (b) Tangents parallel to the x-axis mean maximum and minimum (see graph)

EITHER by GDC P(-2, 29) and Q(4, -79)

OR $f'(x) = 0 \Leftrightarrow 3x^2 - 6x - 24 = 0 \Leftrightarrow x = -2$ or $x = 4$

Coordinates are P(-2, 29) and Q(4, -79)

(c)



- (i) (4, 29) (ii) (-2, -79)

24. $f(x) = 3x^2$

(a) $f'(x) = 6x, f'(a) = 6a$

(b) At $x = a, y = 3a^2, m_T = 6a$

Tangent line: $y - 3a^2 = 6a(x - a) \Rightarrow y - 3a^2 = 6ax - 6a^2$

$\Rightarrow y = 6ax - 3a^2$

(c) $-3a^2 = -3 \Leftrightarrow a^2 = 1 \Leftrightarrow a = \pm 1$

(d) For $a = 1$, the line is $y = 6x - 3$

For $a = -1$, the line is $y = -6x - 3$

25. $f(x) = 2x^3$

(a) $f'(x) = 6x^2$

At $x = 1$, $y = 2$, Point $(1, 2)$

$$m_T = 6$$

Tangent line: $y - 2 = 6(x - 1)$ (i.e. $y = 6x - 4$)

(b) At $x = a$, $y = 2a^3$, Point $(a, 2a^3)$

$$m_T = 6a^2$$

Tangent line: $y - 2a^3 = 6a^2(x - a) \Rightarrow y - 2a^3 = 6a^2x - 6a^3$

$$\Rightarrow y = 6a^2x - 4a^3$$

(c) $-4a^3 = 4 \Leftrightarrow a^3 = -1 \Leftrightarrow a = -1$

Hence, the tangent line is $y = 6x + 4$