INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

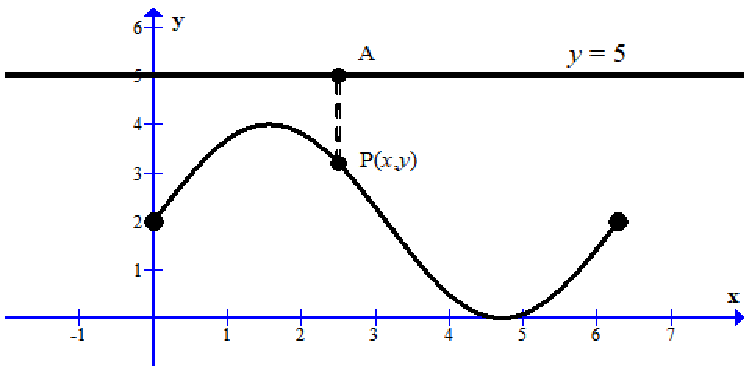
EXERCISES [MAA 5.8]

OPTIMISATION

O. Practice questions

1. [Maximum mark: 7] ***[without GDC]***

The following diagram shows the graph of the function $f(x)=2\sin x+2$, $0\le x\le 2\pi$.



A vertical line segment [AP] is drawn between and the horizontal line $y=5$ and the curve $y=f(x)$, where A is on $y=5$ and P(x,y) on the curve, as shown above.

(a) Explain why the length L of AP is given by $L=3-2\sin x$ [1]

(b) Find the minimum and the maximum value of L by using differentiation on L. [6]

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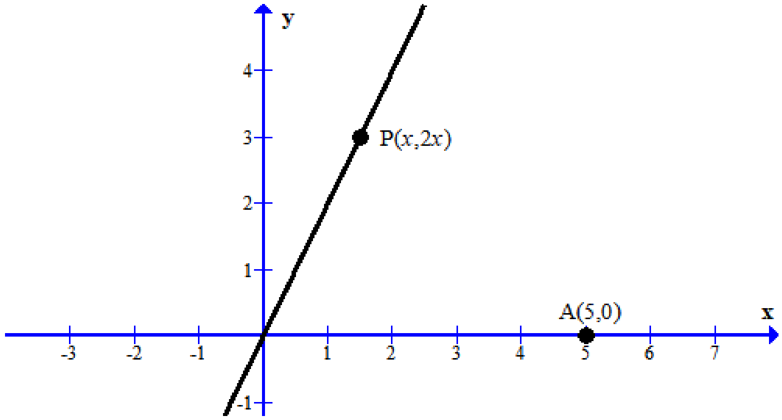
2. [Maximum mark: 8] ***[without GDC]***

A point on the line $y = 2x$ has coordinates P$\left(x, 2x \right)$. Let A(5,0) ..



(a) Express the distance D between P and A in terms of x. [2]

(b) Find $\dfrac{\mathrm{d}D}{\mathrm{d}x}$ [3]

(c) **Hence,** find the minimum distance D and the coordinates of the corresponding point P (i.e. the closest to A). [3]

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3. [Maximum mark: 9] ***[with / without GDC]***

A point on the curve $y = x^2$ has coordinates P$\left(a, a^2\right)$. The point A$\left(2, \frac{1}{2}\right)$ does not lie in the curve.

(a) Express the distance D between P and A in terms of a. [2]

(b) Find $\frac{\mathrm{d}D}{\mathrm{d}a}$ [3]

(c) **Hence** find

(i) the coordinates of the point P on the line which is closest to A;

(ii) the minimum distance D between the point A and the curve $y = x^2$. [4]

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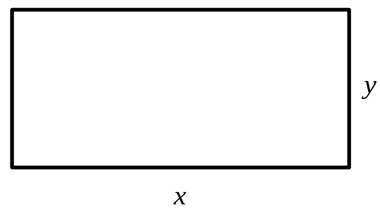
4. [Maximum mark: 8] ***[without GDC]***

A rectangle has sides x and y and its perimeter is 40m.



(a) Express y in term of x. [1]

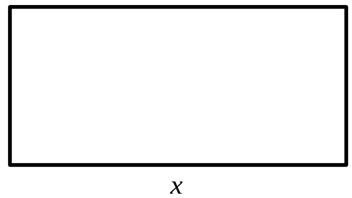
(b) Hence, express the area A of the rectangle in terms of x. [1]

(c) Use $\frac{\mathrm{d}A}{\mathrm{d}x}$ to find the maximum area of the rectangle; justify your answer. [5]

(d) What is the minimum value of A? [1]

5. [Maximum mark: 9] ***[without GDC]***

The area of a rectangle of length x is 100 m^2.



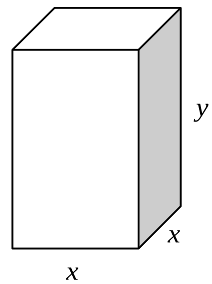
(a) Express the perimeter P of the rectangle in terms of x. [3]

(b) Use $\frac{\mathrm{d}P}{\mathrm{d}x}$ to find the minimum perimeter of the rectangle; justify your answer. [5]

(c) Explain why there is no maximum value for P. [1]

6. [Maximum mark: 9] ***[without GDC]***

The following diagram shows a cuboid of square base of side x and height y. The volume of the cuboid is 1000 cm^2



(a) Express y in term of x. [2]

(b) **Hence**, express the surface area S of the cuboid in terms of x. [2]

(c) Use $\frac{\mathrm{d}S}{\mathrm{d}x}$ to find the minimum value of S; justify your answer. [5]

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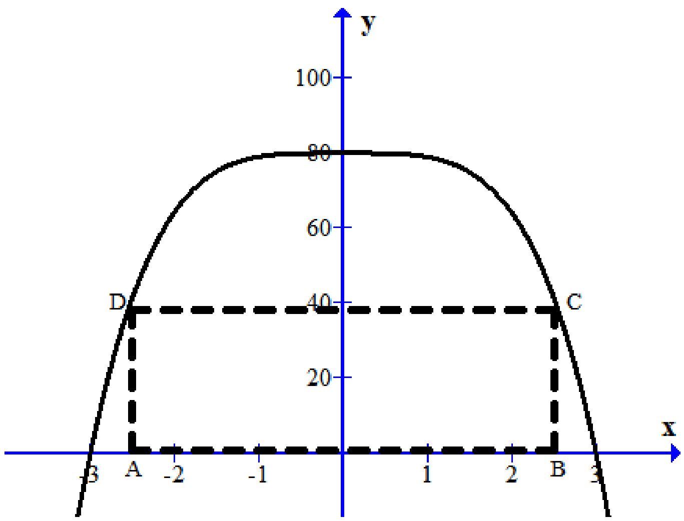
7. [Maximum mark: 7] ***[without GDC]***

Let $f(x) = 80 - x^4$. A rectangle ABCD is inscribed in the region between the curve $y = f(x)$ and x-axis as shown in the diagram below. Assume that he lower vertices of the rectangle have coordinates A(-a,0) and B(a,0)



(a) Show that the area of the rectangle is given by $S = 160a - 2a^5$. [2]

(b) Find the maximum value S, justifying it is a maximum. [5]

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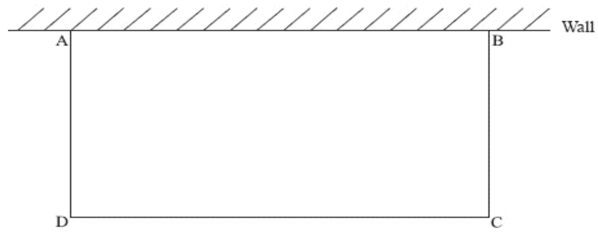
A. Exam style questions (SHORT)

8. [Maximum mark: 7 ***[without GDC]***

The following diagram shows a rectangular area ABCD enclosed on three sides by 60 m of fencing, and on the fourth by a wall AB.



(a) Find the width of the rectangle that gives its maximum area. [6]

(b) Find the maximum area. [1]

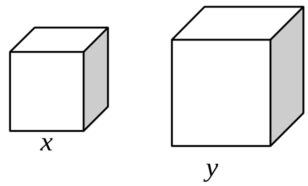
9. [Maximum mark: 7] ***[without GDC]***

A farmer wishes to create a rectangular enclosure, ABCD, of area 525 m^2, as shown below.

The fencing used for side AB costs \$11 per metre. The fencing for the other three sides costs \$3 per metre. The farmer creates an enclosure so that the cost is a minimum. Find this minimum cost.

10. [Maximum mark: 9] ***[with / without GDC]***

Two cubes of edges x and y respectively are shown below.



The total surface area of the two cubes is 300 cm^2.

(a) Find an expression of y in terms of x. [3]

(b) Given that there is a minimum value of the total volume of the cubes, find this minimum value and the corresponding dimensions of the two cubes. [6]

11. [Maximum mark: 8] ***[without GDC]***

A closed cylindrical can has a volume of $250\pi\,\text{cm}^3$. The height of the can is h cm and the radius of the base is r cm.

(a) Find an expression for the total surface area A of the can, in terms of r. [3]

(b) Given that there is a minimum value of A for $r > 0$, find this value of r. [3]

(c) Show that for that can of minimum surface area, the height of the cylinder is equal to the diameter of its base. [2]

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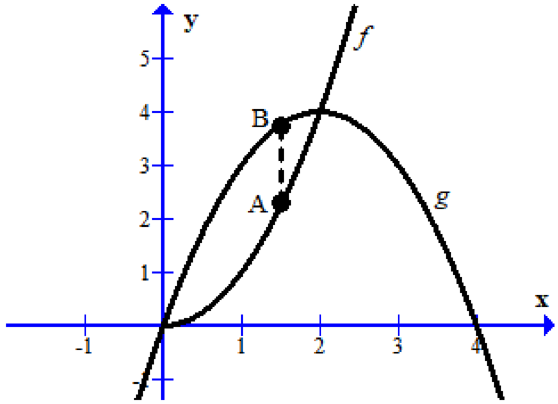
12. [Maximum mark: 9] ***[without GDC]***

The following diagram shows parts of the graphs of $f(x) = x^2$ and $g(x) = 4x - x^2$.



The two graphs intersect at $x = 0$ and $x = a$.

(a) Find the value of a. [3]

A vertical line segment [AB] is drawn between the curves in the interval $0 \le x \le a$ where A is on $y = f(x)$ and B is on $y = g(x)$, as shown above.

(b) Express the length L of the line segment [AB] in terms of x. [1]

(c) Find the maximum value of L, justifying it is a maximum. [5]

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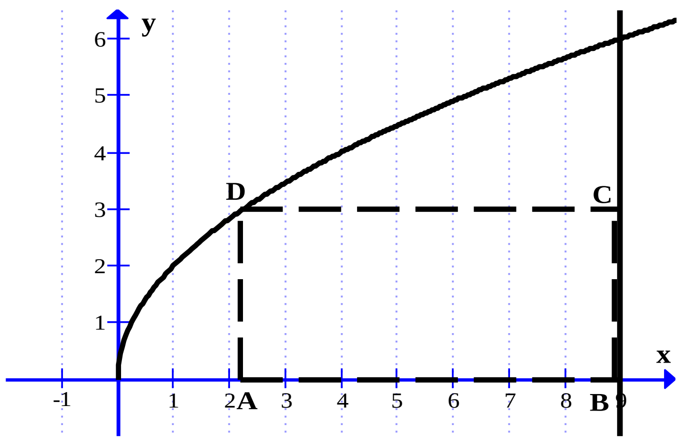
13. [Maximum mark: 9] ***[without GDC]***

A rectangle ABCD is enclosed between the x-axis, the curve $y = 2\sqrt{x}$, and the vertical line $x = 9$, so that the lower vertices are on the x-axis with coordinates A(a,0) and B(9,0) respectively, the vertex D is on the curve, while the vertex C is on the vertical line. The area of this rectangle is denoted by S.



(a) Find an expression of S in terms of a. [2]

(b) Find the maximum value of S; justify your answer. [5]

(c) Write down the values of a for which we obtain a rectangle of minimum area S. [2]

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14. [Maximum mark: 5] ***[with GDC]***

A point $\mathrm{P}(x, x^2)$ lies on the curve $y = x^2$.

(a) Calculate the minimum distance from the point $\mathrm{A}\left(2, -\frac{1}{2}\right)$ to the point P. [4]

(b) Write down the coordinates of the closest point P to the point A. [1]

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15. [Maximum mark: 6] ***[with GDC]***

The point $\mathrm{B}(a,b)$ is on the curve $f(x) = x^2$ such that B is the point which is closest to $\mathrm{A}(6,0)$.

(a) Show that $2a^3 + a - 6 = 0$. [5]

(b) **Hence**, or otherwise, find the value of a. [1]

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16. [Maximum mark: 4] ***[with GDC]***

A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = \sin x$, where $0 \le x \le \pi$.

(a) Write down an expression for the area of the rectangle.

(b) Find the maximum area of the rectangle.

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17. [Maximum mark: 6] ***[with / without GDC]***

A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = \mathrm{e}^{-x^2}$. The area of this rectangle is denoted by A.

(a) Write down an expression for A in terms of x.

(b) Find the maximum value of A.

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18. [Maximum mark: 8] ***[without GDC]***

A packaging company makes boxes for chocolates. An example of a box is shown below. This box is closed and the top and bottom of the box are identical regular hexagons of side x cm.

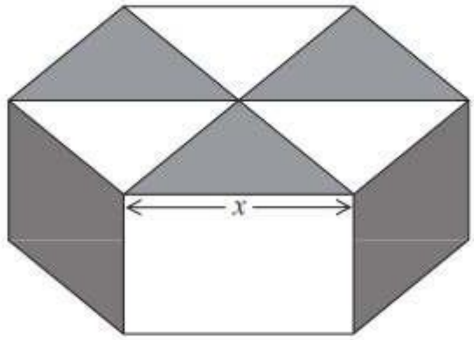

diagram not to scale

(a) Show that the area of each hexagon is $\frac{3\sqrt{3}x}{2}$ cm^2. [2]

(a) Given that the volume of the box is $90\ \text{cm}^3$, show that when $x = \sqrt[3]{20}$ cm the total surface area of the box is a minimum, justifying that this value gives a minimum. [6]

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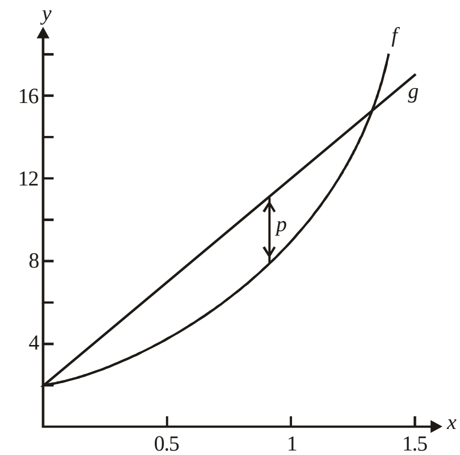
B. Exam style questions (LONG)

19. [Maximum mark: 10] ***[without GDC]***

The diagram below shows the graphs of $f(x) = 1 + e^{2x}$, $g(x) = 10x + 2$, $0 \le x \le 1.5$.



(a) Find the coordinates of the two points of intersection of the two graphs. [2]

(b) Let p be the vertical distance between the graphs of f and g for $0 \le x \le 1.3$.

(i) Write down an expression for p in terms of x.

(ii) Find the **exact** value of x at which the maximum value of p occurs. Justify your answer.

(iii) Hence find the maximum value of p. [8]

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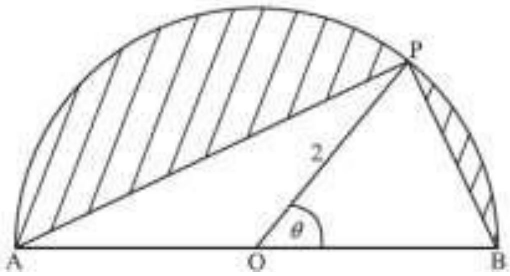
20. [Maximum mark: 18] ***[without GDC]***

The following diagram shows a semicircle centre O, diameter [AB], with radius 2. Let P be a point on the circumference, with $P\hat{O}B = \theta$ radians.



(a) Find the area of the triangle OPB, in terms of θ. [2]

(b) Explain why the area of the triangle OPA is the same as the area triangle OPB. [3]

Let S be the total area of the two segments shaded in the diagram above.

(c) Show that $S = 2(\pi - 2\sin\theta)$. [3]

(d) Find the value of θ when S is a local minimum, justifying it is a minimum. [8]

(e) Find a value of θ for which S has its greatest value. [2]

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21*. [Maximum mark: 9] ***[with / without GDC]***

Consider the graph *G* of the function $y = \sqrt{x}$ and its reflection *G´* about the line $x = 4$.

(a) Sketch the graphs of *G* and *G´*; indicate the coordinates of the intersection point of the two graphs. [2]

A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices on the curves *G* and *G´* respectively. Let x be the x-coordinate of the lower left corner of the rectangle.

(b) Show that an expression for the area of the rectangle is given by $S = 8\sqrt{x} - 2x^{3/2}$. [2]

(c) Find the maximum value of S. Justify your answer. [5]

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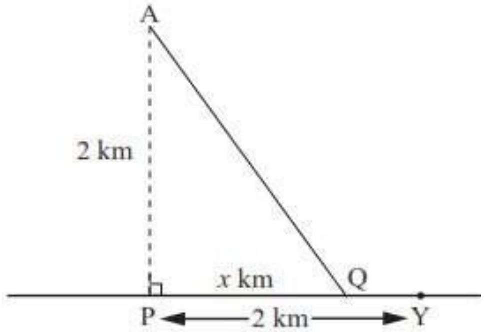
22*. [Maximum mark: 18] ***[without GDC]***

André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that $\mathrm{AP} = 2\ \mathrm{km}$ and $\mathrm{PY} = 2\ \mathrm{km}$. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers $1\ \mathrm{km}$ in $5\sqrt{5}$ minutes. When he runs he covers $1\ \mathrm{km}$ in 5 minutes.

(a) If $\mathrm{PQ} = x\ \mathrm{km}$, $0 \le x \le 2$, find an expression for the time T minutes taken by André to reach point Y. [4]

(b) Show that $\dfrac{\mathrm{d}T}{\mathrm{d}x} = \dfrac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$. [3]

(c) (i) Solve $\dfrac{\mathrm{d}T}{\mathrm{d}x} = 0$.

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\dfrac{\mathrm{d}^2T}{\mathrm{d}x^2} = \dfrac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum. [11]

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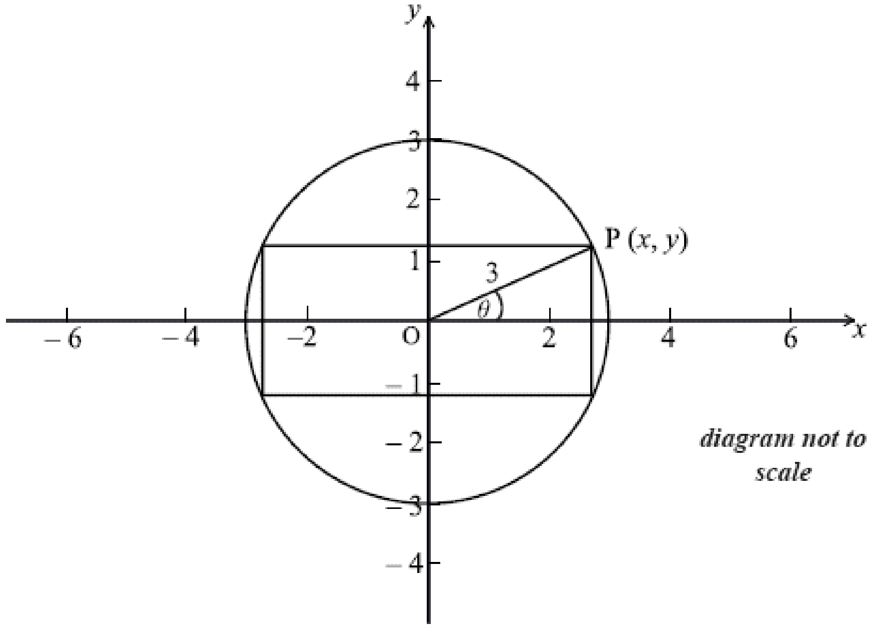
23*. [Maximum mark: 13] ***[without GDC]***

A rectangle is inscribed in a circle of radius 3 cm and centre O, as shown below.



The point $P(x, y)$ is a vertex of the rectangle and also lies on the circle. The angle between (OP) and the x-axis is θ radians, where $0 \le \theta \le \frac{\pi}{2}$.

(a) Write down an expression in terms of θ for

(i) x;

(ii) y. [2]

Let the area of the rectangle be A.

(b) Show that $A = 18 \sin 2\theta$. [3]

(c) (i) Find $\frac{dA}{d\theta}$.

(ii) Hence, find the exact value of θ which maximizes the area of the rectangle.

(iii) Use the second derivative to justify that this value of θ does give a maximum. [8]

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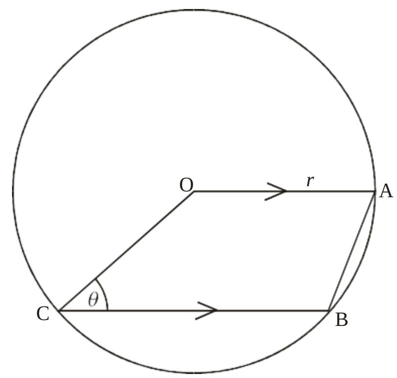
24. [Maximum mark: 15] ***[with GDC]***

The diagram shows a trapezium OABC in which OA is parallel to CB. O is the centre of a circle radius r cm. A, B and C are on its circumference. Angle $\mathrm{OCB} = \theta$.



Let T denote the area of the trapezium OABC.

(a) Show that $T = \frac{r^2}{2}(\sin\theta + \sin 2\theta)$. [4]

For a **fixed** value of r, the value of T varies as the value of θ varies.

(b) Show that T takes its maximum value when θ satisfies the equation $4\cos^2\theta + \cos\theta - 2 = 0$, and verify that this value of T is a maximum. [5]

(c) Given that the perimeter of the trapezium is 75 cm, find the maximum value of T. [6]

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25*. [Maximum mark: 15] ***[without GDC]***

A **square** of side a has perimeter $P = 4a$ and area $A = a^2$.

(a) Consider all rectangles of constant perimeter $P = 4a$.

(i) Given that x is one side of the rectangle, show its area A is given by $A = 2ax - x^2$

(ii) Given that one side of the rectangle is $\frac{a}{2}$, find its area.

(iii) Show that the rectangle of maximum area is the square, with $A_{max} = a^2$. Justify your answer. [7]

(b) Consider all rectangles of constant area $A = a^2$.

(i) Given that x is one side of the rectangle, find an expression for its perimeter P in terms of a and x.

(ii) Given that one side of the rectangle is $\frac{a}{2}$, find its perimeter.

(iii) Show that the rectangle of minimum perimeter is the square, with $P_{min} = 4a$. Justify your answer. [8]

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26*. [Maximum mark: 12] ***[without GDC]***

Consider all closed cylinders of height h, base radius r and constant volume V.

(a) Express the total surface area S of the cylinder in terms of V and r. [4]

(b) Find $\frac{\mathrm{d}S}{\mathrm{d}r}$ [2]

(c) Hence show that the radius of the cylinder of minimum surface area is $r=\sqrt[3]{\frac{V}{2\pi}}$.

Justify that it is a minimum. [4]

(d) Do there exist cylinders of maximum total surface area? [2]

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27*. [Maximum mark: 12] ***[without GDC]***

Consider all closed cylinders of height h, base radius r and constant surface area S.

(a) Express the volume V of the cylinder in terms of S and r. [4]

(b) Find $\frac{\mathrm{d}V}{\mathrm{d}r}$ [2]

(c) Hence show that the radius of the cylinder of maximum volume is $r = \sqrt{\frac{S}{6\pi}}$.

Justify that it is a maximum . [4]

(b) Do there exist cylinders of minimum volume? [2]

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