[MAA 5.4] CHAIN RULE

SOLUTIONS

O. Practice questions

1. Solutions are shown in the question.

2.

$y = f(x)$	$\dfrac{dy}{dx} = f'(x)$
$y = e^{2x} + \ln(2x+3)$	$\dfrac{dy}{dx} = 2e^{2x} + \dfrac{2}{2x+3}$
$y = e^{2\pi} + \ln(2\pi+3)$	$\dfrac{dy}{dx} = 0$ (since y is a constant)
$y = \sqrt{x^5 + 2x + 1}$	$\dfrac{dy}{dx} = \dfrac{5x^4 + 2}{2\sqrt{x^5 + 2x + 1}}$
$y = \dfrac{2x+1}{3x-5}$	$\dfrac{dy}{dx} = \dfrac{2(3x-5) - 3(2x+1)}{(3x-5)^2} = \dfrac{-13}{(3x-5)^2}$
$y = \dfrac{2x+1}{(3x-5)^2} = (2x+1)(3x-5)^{-2}$	$\dfrac{dy}{dx} = 2(3x-5)^{-2} - 6(2x+1)(3x-5)^{-3}$
$y = \ln\dfrac{2x+1}{(3x-5)^2} = \ln(2x+1) - 2\ln(3x-5)$	$\dfrac{dy}{dx} = \dfrac{2}{2x+1} - \dfrac{6}{3x-5}$
$y = \ln\dfrac{(2x+1)^3}{(3x-5)^2} = 3\ln(2x+1) - 2\ln(3x-5)$	$\dfrac{dy}{dx} = \dfrac{6}{2x+1} - \dfrac{6}{3x-5}$
$y = \ln\dfrac{\sqrt{x+3}}{(x+1)^3} = \dfrac{1}{2}\ln(x+3) - 3\ln(x+1)$	$\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{1}{x+3} - \dfrac{3}{x+1}$

3.

$s = 10\cos t$	$\dfrac{ds}{dt} = -10\sin t$
$s = \cos(10t)$	$\dfrac{ds}{dt} = -10\sin 10t$
$s = \cos(t^{10})$	$\dfrac{ds}{dt} = -10t^9 \sin(t^{10})$
$s = \cos^{10} t$	$\dfrac{ds}{dt} = -10(\cos^9 t)(\sin t)$

4.

Function: y	**Derivative:** $\dfrac{dy}{dx}$
$y = f(x)^3$	$\dfrac{dy}{dx} = 3f(x)^2 f'(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = e^{f(x)} f'(x)$
$y = \sin f(x)$	$\dfrac{dy}{dx} = \cos f(x) \times f'(x)$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = \sqrt{f(x)}$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{2\sqrt{f(x)}}$

Function: y	**Derivative:** $\dfrac{dy}{dx}$
$y = f(x^3)$	$\dfrac{dy}{dx} = 3x^2 f'(x^3)$
$y = f(e^x)$	$\dfrac{dy}{dx} = f(e^x) \times e^x$
$y = f(\sin x)$	$\dfrac{dy}{dx} = f(\sin x) \times \cos x$
$y = f(\ln x)$	$\dfrac{dy}{dx} = \dfrac{f(\ln x)}{x}$
$y = f(\sqrt{x})$	$\dfrac{dy}{dx} = \dfrac{f(\sqrt{x})}{2\sqrt{x}}$

5.

Function	**Derivative**	**Derivative at** $x = 1$
$f(x)^3$	$3f(x)^2 f'(x)$	$3f(1)^2 f'(1) = 3\cdot(-2)^2 \cdot 3 = 36$
$f(2x-2)$	$2f'(2x-2)$	$2f'(0) = 2$
$e^{f(x)}$	$e^{f(x)} f'(x)$	$e^{f(1)} f'(1) = 3e^{-2} = \dfrac{3}{e^2}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$	$\dfrac{f'(1)}{f(1)} = -\dfrac{3}{2}$
$f(\ln x)$	$\dfrac{f'(\ln x)}{x}$	$\dfrac{f'(\ln 1)}{1} = f'(0) = 1$
$f(-g(x))$	$-f'(-g(x))\,g'(x)$	$-f'(-g(1))\,g'(1) = -f'(1)\,g'(1) = -6$
$g(f(x)+2)$	$g'(f(x)+2)\,f'(x)$	$g'(f(1)+2)\,f'(1) = g'(0)\,f'(1) = 15$

6. (a) $y = \tan x = \dfrac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\sin x \sin x - \cos x(-\cos x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

(b) (i) $\dfrac{dy}{dx} = \dfrac{x}{\cos^2 x} + \tan x$

(ii) $\dfrac{dy}{dx} = \dfrac{3}{\cos^2 3x}$

(iii) $\dfrac{dy}{dx} = \dfrac{2\tan x}{\cos^2 x} \quad \left(= \dfrac{2\sin x}{\cos^3 x} \right)$

(iv) $\dfrac{dy}{dx} = \dfrac{3\tan^2 x}{\cos^2 x} \quad \left(= \dfrac{3\sin^2 x}{\cos^4 x} \right)$

A. Exam style questions (SHORT)

7. (a) $12x(x^2+5)^2$ (b) $4xe^{x^2+1}$

8. (a) $-7e^{-x}+4e^{\frac{x}{2}}$ (b) $-9\sin 3x+1$

9. (a) $\frac{2x}{x^2+1}-\pi\sin\left(\frac{\pi}{2}x\right)$ (b) $\frac{x}{\sqrt{x^2+5}}+\frac{1}{3}x^{-2/3}$

10. (a) $6(2x+5)^2+8x+22$

(b) $f(x)=1$, so $0\ f'(x)=0$

OR $f'(x)=2\sin x\cos x-2\cos x\sin x=0$

11. (i) $\frac{dy}{dx}=\sin 3x+3x\cos 3x$

(i) $\frac{dy}{dx}=\frac{x\times\frac{1}{x}-\ln x}{x^2}=\frac{1-\ln x}{x^2}$

12. (a) $\frac{d}{dx}(x^2+1)^2=2(x^2+1)\times(2x)=4x(x^2+1)$

(b) $\frac{d}{dx}(\ln(3x-1))=\frac{1}{3x-1}\times(3)=\frac{3}{3x-1}$

13. (a) $\frac{dy}{dx}=\frac{-4}{2\sqrt{3-4x}}=\frac{-2}{\sqrt{3-4x}}$

OR $y=\sqrt{3-4x}=(3-4x)^{\frac{1}{2}}$ $\frac{dy}{dx}=\frac{1}{2}(3-4x)^{-\frac{1}{2}}(-4)$

(b) $y=e^{\sin x}$ $\frac{dy}{dx}=(\cos x)(e^{\sin x})$

14. $f'(x)=\frac{1}{3}e^{\frac{x}{3}}-10\cos x\sin x$

15. (a) $f'(x)=3(2x+7)^2\times 2=6(2x+7)^2$ $(=24x^2+168x+294)$

(b) $g'(x)=2\cos(4x)(-\sin(4x))(4)=-8\cos(4x)\sin(4x)$ $(=-4\sin(8x))$

16. $y=\ln(2x-1)\Rightarrow\frac{dy}{dx}=\frac{2}{2x-1}\Rightarrow\frac{dy}{dx}=2(2x-1)^{-1}$

$\Rightarrow\frac{d^2y}{dx^2}=-2(2x-1)^{-2}(2)$

$\Rightarrow\frac{d^2y}{dx^2}=\frac{-4}{(2x-1)^2}$

17. (a) If $f(x)=\ln(2x-1)$, Then $f'(x)=\frac{2}{2x-1}$

(b) Put $\frac{2}{2x-1}=x\Rightarrow x=1.28$ (using a GDC or the quadratic formula)

18. $y = \sin(2x - 1)$ $\quad \dfrac{dy}{dx} = 2\cos(2x - 1)$

At $\left(\dfrac{1}{2}, 0\right)$, the gradient of the tangent = 2 cos 0 = 2

19. $y = \sin(kx) - kx\cos(kx)$

$\dfrac{dy}{dx} = k\cos(kx) - k\{\cos(kx) + x[-k\sin(kx)]\}$

$= k\cos(kx) - k\cos(kx) + k^2x\sin(kx) = k^2x\sin(kx)$

20.

(a) **METHOD 1**

$f(x) = \dfrac{1}{2}\ln(x^2 + 4)$

$f'(x) = \dfrac{1}{2}\left(\dfrac{2x}{x^2 + 4}\right)$

$f'(x) = \dfrac{x}{x^2 + 4}$

METHOD 2

$f'(x) = \dfrac{1}{\sqrt{x^2 + 4}} \times \dfrac{1}{2}(x^2 + 4)^{\frac{-1}{2}}(2x)$

$= \dfrac{x}{x^2 + 4}$

(b) $f''(x) = \dfrac{x^2 + 4 - x(2x)}{(x^2 + 4)^2}$

$= \dfrac{4 - x^2}{(x^2 + 4)^2}$

21. (a) $f'(x) = -\sin 2x \times 2 \; (= -2\sin 2x)$

(b) $g'(x) = 3 \times \dfrac{1}{3x - 5}\left(= \dfrac{3}{3x - 5}\right)$

(c) product rule: $h'(x) = (\cos 2x)\left(\dfrac{3}{3x - 5}\right) + \ln(3x - 5)(-2\sin 2x)$

22. (a) $f'(x) = 5e^{5x}$

(b) $g'(x) = 2\cos 2x$

(c) $h' = fg' + gf' = e^{5x}(2\cos 2x) + \sin 2x\,(5e^{5x})$

23. (a) (i) $-3e^{-3x}$ (ii) $\cos\left(x - \dfrac{\pi}{3}\right)$

(b) product rule

$h'(x) = -3e^{-3x}\sin\left(x - \dfrac{\pi}{3}\right) + e^{-3x}\cos\left(x - \dfrac{\pi}{3}\right)$

$h'\left(\dfrac{\pi}{3}\right) = -3e^{-3\frac{\pi}{3}}\sin\left(\dfrac{\pi}{3} - \dfrac{\pi}{3}\right) + e^{-3\frac{\pi}{3}}\cos\left(\dfrac{\pi}{3} - \dfrac{\pi}{3}\right) = e^{-\pi}$

24. (a) $p = 100e^0 = 100$

(b) Rate of increase is $\dfrac{dp}{dt} = 0.05 \times 100e^{0.05t} = 5e^{0.05t}$

When $t = 10$ $\quad \dfrac{dp}{dt} = 5e^{0.05(10)} = 5e^{0.5} \quad \left(= 8.24 = 5\sqrt{e}\right)$

25. (a) $n = 800e^0 = 800$

(b) derivative: $n'(15) = 731$

(c) **METHOD 1**
setting up inequality. $n'(t) > 10\,000$

$k = 35.1226...$
least value of k is 36

METHOD 2
$n'(35) = 9842$, **and** $n'(36) = 11208$
least value of k is 36

26. (a) $1300 = 650e^{20k} \Leftrightarrow k = \dfrac{\ln 2}{20}$

(b) $\dfrac{dn}{dt} = 650ke^{kt}$

when $t = 90$, $\dfrac{dn}{dt} = 509.734 = 510$ to 3 s.f.

27. (i) $y = f(x)^2 \Rightarrow \dfrac{dy}{dx} = 2f(x)f'(x)$. At $x = 1$, $\dfrac{dy}{dx} = 2\times2\times4 = 16$

(ii) $y = f(x^2) \Rightarrow \dfrac{dy}{dx} = f'(x^2)\times 2x$. At $x = 1$, $\dfrac{dy}{dx} = 2\times2\times1 = 4$

28. (i) $y = f(x)^3 \Rightarrow \dfrac{dy}{dx} = 3f(x)^2 f'(x)$. At $x = 1$, $\dfrac{dy}{dx} = 3\times2^2\times4 = 48$

(ii) $y = f(x^3) \Rightarrow \dfrac{dy}{dx} = f'(x^3)\times 3x^2$. At $x = 1$, $\dfrac{dy}{dx} = 2\times3\times1^2 = 6$

29. (i) $y = \ln f(x) \Rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$. At $x = 1$, $\dfrac{dy}{dx} = \dfrac{4}{2} = 2$

(ii) $y = \sqrt{f(x)} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{f(x)}}f'(x)$. At $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{2}}\times4 = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

30.

(a) derivative $= \dfrac{3f'(x)[g(x)-1]-3f(x)g'(x)}{[g(x)-1]^2}$

when $x = 0 \Rightarrow$ derivative $= \dfrac{3(1)(-4-1)-3(4)(5)}{(-4-1)^2} = -3$

(b) derivative $= f'(g(x)+2x)(g'(x)+2)$
when $x = 1$ derivative $= f'(-1+2)(2+2) = (3)(4) = 12$

31. (a) $f(x) = \dfrac{1}{2}\sin 2x + \cos x$
$f'(x) = \cos 2x - \sin x$
$= 1 - 2\sin^2 x - \sin x = (1 + \sin x)(1 - 2\sin x)$
$= 0$ when $\sin x = -1$ or $\dfrac{1}{2}$

(b) $x = -\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

32. $y = e^{3x}\sin(\pi x)$

(a) $\dfrac{dy}{dx} = 3e^{3x}\sin(\pi x) + \pi e^{3x}\cos(\pi x)$

(b) $0 = e^{3x}(3\sin(\pi x) + \pi\cos(\pi x)) \Rightarrow \tan(\pi x) = -\dfrac{\pi}{3}$

$\pi x = -0.80845 + \pi \Rightarrow x = 0.7426\ldots$ (0.743 to 3 s.f.)

33.

(a) $f'(x) = 3\cos^2(4x+1)\times(-\sin(4x+1))\times 4$

$f'(x) = -12\cos^2(4x+1)(\sin(4x+1))$

(b) $f'(x) = 0 \Rightarrow \cos^2(4x+1) = 0$ **or** $\sin(4x+1) = 0$

$\Rightarrow x = \dfrac{\pi}{8} - \dfrac{1}{4},\ x = \dfrac{3\pi}{8} - \dfrac{1}{4}$ **or** $x = \dfrac{\pi - 1}{4}$

34. (a) $f'(x) = \dfrac{(x+2)(2x+5) - (x^2+5x+5)}{(x+2)^2} = \dfrac{x^2+4x+5}{(x+2)^2}$

(b) $\dfrac{x^2+4x+5}{(x+2)^2} > 2$

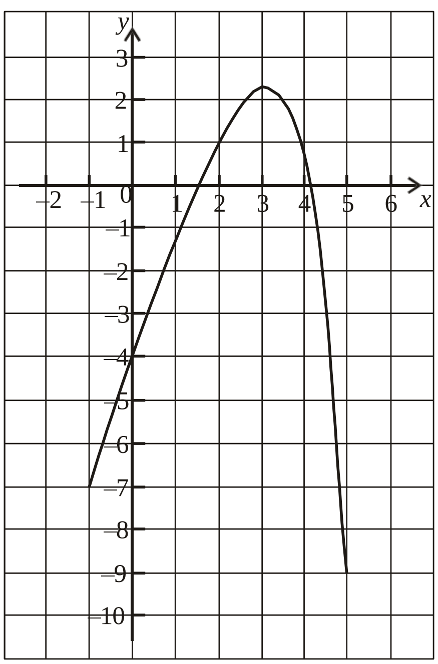
$\Rightarrow x^2 + 4x + 5 > 2x^2 + 8x + 8$

$\Rightarrow x^2 + 4x + 3 < 0 \Rightarrow -3 < x < -1$ (and $x \neq -2$)

(NOTICE: by a graph in GDC, we can find the same solution)

35. (a) (1.54, 0) (4.13, 0) (accept $x = 1.54$ $x = 4.13$)

(b)



Note: *Curve passing through (0, − 4), a range of approximately −9 to 2.3.*

(c) gradient is 2

B. Exam style questions (LONG)

36. (a) (i) $f'(x) = -6\sin 2x$

(ii) **EITHER** $f'(x) = -12\sin x\cos x = 0 \Rightarrow \sin x = 0$ or $\cos x = 0$

OR $\sin 2x = 0$, for $0 \le 2x \le 2\pi$

THEN $x = 0, \frac{\pi}{2}, \pi$

(b) (i) translation in the y-direction of -1

(ii) 1.11 (1.10 from TRACE is subject to **AP**)

37. (a) recognizing the amplitude is the radius: $a = \frac{8}{2} \Rightarrow a = 4$

(b) period = 30: $b = \frac{2\pi}{30} = \frac{\pi}{15}$

(c) recognizing $h'(t) = -0.5$

$-0.5 = \frac{4\pi}{15}\cos\left(\frac{\pi}{15}t\right) \Rightarrow t = 10.6,\ t = 19.4$

(d) $h(t) < 0$ so underwater; $h(t) > 0$ so not underwater

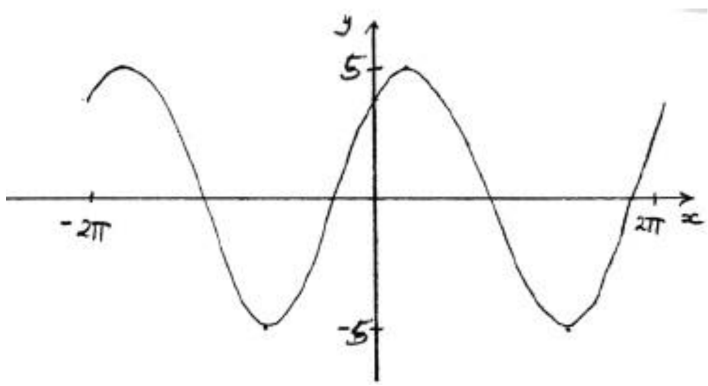
$h(19.4) = 4\sin\frac{19.4\pi}{15} + 2 = -1.19$

OR

solving $h(t) = 0$, graph showing region below x-axis, roots 17.5, 27.5

Hence, the bucket is underwater, yes

38. (a)



Note: *approximately sinusoidal shape,*
end points approximately correct, (–2π, 4), (2π, 4)
approximately correct position of graph,
(y-intercept (0, 4) maximum to right of y-axis).

(b) (i) 5 (ii) 2π (6.28) (iii) -0.927

(c) $f(x) = 5\sin(x + 0.927)$ (accept $p = 5$, $q = 1$, $r = 0.927$)

(d) (man or min)

one 3 s.f. value which rounds to one of –5.6, –2.5, 0.64, 3.8

(e) $k = -5$, $k = 5$

(f) $g'(x) = \frac{1}{x+1}$

$f'(x) = 3\cos x - 4\sin x$ $(5\cos(x + 0.927))$

$g'(x) = f'(x)$

$x = 0.511$