INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 5.23]

MACLAURIN SERIES – EXTENSION OF BINOMIAL THEOREM

O. Practice questions

1. [Maximum mark: 7] ***[without GDC]***

Find the Maclaurin series of the function $f(x) = e^{2x}$ up to and including the term in x^4

(a) by using the formula of the Maclaurin series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ [5]

(b) by using the Maclaurin series of $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ [2]

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2. [Maximum mark: 8] ***[without GDC]***

Find the Maclaurin series of the function $f(x) = (2+x)^3$

(a) by using the formula of the Maclaurin series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ [5]

(b) by using the expansion of the binomial theorem. [3]

3. [Maximum mark: 10] ***[without GDC]***

Find the Maclaurin series of the function $f(x) = \frac{1}{(2+x)^2} = (2+x)^{-2}$, up to x^2

(a) By using the formula of the Maclaurin series. [4]

(b) By using the extended version of the binomial theorem. [4]

(c) Write down the values of x for which the series converges. [2]

4. [Maximum mark: 9] ***[without GDC]***

Find the Maclaurin series of the function $f(x) = \sqrt{2+x} = (2+x)^{\frac{1}{2}}$, up to x^2

(a) by using the formula of the Maclaurin series. [5]

(b) by using the extended version of the binomial theorem. [4]

5. [Maximum mark: 8] ***[without GDC]***

(a) Find the Maclaurin series of the function $f(x) = \sqrt{x^2 + 1}$, up to and including the term in x^6 by using the extended version of the binomial theorem. [4]

(b) Given that $f'(x) = \frac{x}{\sqrt{x^2 + 1}}$ find the Maclaurin series of $g(x) = \frac{x}{\sqrt{x^2 + 1}}$ up to and including the term in x^5. [4]

6. [Maximum mark: 17] ***[without GDC]***

By using the Maclaurin series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, find the Maclaurin series of

(a) $f(x) = e^{-x}$ up to and including the term in x^3. [2]

(b) $f(x) = e^{-x^2}$ up to and including the term in x^6. [3]

(c) $f(x) = xe^x$ up to and including the term in x^4. [2]

(d) $f(x) = x^2e^{-x}$ up to and including the term in x^5. [3]

(e) $f(x) = e^x - e^{-x}$ up to and including the term in x^5. [3]

(f) $f(x) = (x+1)e^{4x}$ up to and including the term in x^3. [4]

A. Exam style questions (SHORT)

7. [Maximum mark: 10] ***[without GDC]***

(a) Find the first three terms of the Maclaurin series for $\ln(1+e^x)$. [6]

(b) Hence, or otherwise, determine the value of $\lim_{x\to 0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2}$. [4]

8. [Maximum mark: 10] ***[without GDC]***

The function f is defined by $f(x) = \ln\left(\frac{1}{1-x}\right)$.

(a) Write down the value of the constant term in the Maclaurin series for $f(x)$. [1]

(b) Find the first three derivatives of $f(x)$ and hence show that the Maclaurin series for $f(x)$ up to and including the x^3 term is $x + \frac{x^2}{2} + \frac{x^3}{3}$. [6]

(c) Use this series to find an approximate value for $\ln 2$. [3]

9. [Maximum mark: 10] ***[without GDC]***

Let $f(x) = \ln(1+\sin x)$.

(a) Show that $f'(x) = \dfrac{-1}{1+\sin x}$

(b) Find the third and the fourth derivatives of f

(c) Hence, find the Maclaurin series, up to the term in x^4, for $f(x)$.

10. [Maximum mark: 7] ***[without GDC]***

The variables x and y are related by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y\tan x = \cos x$.

(a) Find the Maclaurin series for y up to and including the term in x^2 given that $y = -\frac{\pi}{2}$ when $x = 0$.

(b) Show that an approximation for y when $x = 0.1$ is $y \cong 0.1 - \frac{201\pi}{400}$

11. [Maximum mark: 7] ***[without GDC]***

(a) Using the extended version of the binomial theorem, find the Maclaurin series of

$$f(x) = \frac{1}{3x+5}$$

up to the term in x^3; [5]

(b) Find the values of x for which the series converges. [2]

12. [Maximum mark: 8] ***[without GDC]***

(a) Using the extended version of the binomial theorem, find the Maclaurin series of

$$f(x) = \sqrt{3x+5}$$

up to the term in x^3; [6]

(b) Find the values of x for which the series converges. [2]

B. Exam style questions (LONG)

13*. [Maximum mark: 22] ***[without GDC]***

It is given that the Maclaurin series of the function $f(x) = \ln(1+\sin x)$, up to the term in x^4 is

$$f(x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$$

(a) Deduce the Maclaurin series, up to and including the term in x^4 for

(i) $y = \ln(1-\sin x)$;

(ii) $y = \ln\cos x$;

(iii) $y = \ln\sec x$; [8]

(b) By differentiating the Maclaurin series of $y = \ln\cos x$, deduce the Maclaurin series of $y = \tan x$ [4]

(c) Hence calculate the limits (i) $\lim\limits_{x\to 0}\dfrac{\ln\sec x}{x\sqrt{x}}$ (ii) $\lim\limits_{x\to 0}\left(\dfrac{\tan(x^2)}{\ln\cos x}\right)$ [6]

(d) By considering the difference of the two series of

$y = \ln(1+\sin x)$ and $y = \ln(1-\sin x)$

deduce that $\ln 3 \approx \dfrac{\pi}{3}\left(1+\dfrac{\pi^2}{216}\right)$. [4]

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14. ([Maximum mark: 14] ***[with GDC]***

(a) Given that $y = \ln \cos x$, show that the first two non-zero terms of the Maclaurin series for y are $-\frac{x^2}{2} - \frac{x^4}{12}$. [8]

(b) Use this series to find an approximation in terms of π for $\ln 2$. [6]