EXERCISES [MAA 5.21] FURTHER AREAS AND VOLUMES

SOLUTIONS

O. Practice questions

1. (a) Using integration by parts

(i) $\int \ln x\mathrm{d}x = \int 1\ln x\mathrm{d}x = x\ln x - \int 1\mathrm{d}x = x\ln x - x + c$

(ii) $\int (\ln x)^2 \mathrm{d}x = x(\ln x)^2 - 2\int \ln x\mathrm{d}x = x(\ln x)^2 - 2x\ln x + 2x + c$

(b) $A = \int_1^{\mathrm{e}} \ln x\mathrm{d}x = [x\ln x - x]_1^{\mathrm{e}} = (\mathrm{e}-\mathrm{e}) - (0-1) = 1$

(c) $A = \pi\int_1^{\mathrm{e}} (\ln x)^2\mathrm{d}x = \pi\left[x(\ln x)^2 - 2x\ln x + 2x\right]_1^{\mathrm{e}} = [(\mathrm{e}-2\mathrm{e}+2\mathrm{e}) - (0-0+2)]\pi$

$= (\mathrm{e}-2)\pi$

2. (a) $A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\mathrm{d}x = [\sin x + \cos x]_0^{\frac{\pi}{4}}$

$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0) = \sqrt{2} - 1$

(b) $A = \int_0^{\frac{\pi}{4}} \cos x\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\mathrm{d}x = [\sin x]_0^{\frac{\pi}{4}} - [\cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left(\frac{\sqrt{2}}{2} - 0\right) - \left(0 - \frac{\sqrt{2}}{2}\right) = \sqrt{2}$

(c) $V = \pi\int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\mathrm{d}x = \pi\int_0^{\frac{\pi}{4}} \cos 2x\mathrm{d}x = \pi\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{\pi}{2}$

(d) $V = \pi\int_0^{\frac{\pi}{4}} \cos^2 x\mathrm{d}x + \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\mathrm{d}x$

3. (a)

Area $A = \frac{b\times h}{2}$	**Volume** $V_x = \frac{1}{3}\pi r^2 h$	**Volume** $V_y = \frac{1}{3}\pi r^2 h$
$A = \frac{4\times 8}{2} = 16$	$V_x = \frac{1}{3}\pi 8^2 \times 4 = \frac{256\pi}{3}$	$V_y = \frac{1}{3}\pi 4^2 \times 8 = \frac{128\pi}{3}$

(b)

Area A **using** $A_x = \int_a^b y\mathrm{d}x$	**Area** A **using** $A_y = \int_a^b x\mathrm{d}y$	**Volume** $V_x = \pi\int_a^b y^2\mathrm{d}x$	**Volume** $V_y = \pi\int_a^b x^2\mathrm{d}y$
$A_x = \int_0^4 (8-2x)\mathrm{d}x$	$A_y = \int_0^8 \frac{8-y}{2}\mathrm{d}y$	$V_x = \pi\int_0^4 (8-2x)^2\mathrm{d}x$	$V_y = \pi\int_0^8 \left(\frac{8-y}{2}\right)^2\mathrm{d}y$

4. (a)

Area	Integral expression	Value
A using $A_x = \int_a^b y\,\mathrm{d}x$	$A_x = \int_0^2 x^2\mathrm{d}x + \int_2^{2\sqrt{2}} \left(8-x^2\right)\mathrm{d}x$	
A using $A_y = \int_a^b x\,\mathrm{d}y$	$A_y = \int_0^4 \left(\sqrt{8-y} - \sqrt{y}\right)\mathrm{d}y$	
B using $A_x = \int_a^b y\,\mathrm{d}x$	$A_x = \int_0^2 \left((8-x^2)-x^2\right)\mathrm{d}x = \int_0^2 (8-2x^2)\mathrm{d}x$	
B using $A_y = \int_a^b x\,\mathrm{d}y$	$A_y = \int_0^4 \sqrt{y}\,\mathrm{d}y + \int_4^8 \sqrt{8-y}\,\mathrm{d}y$	

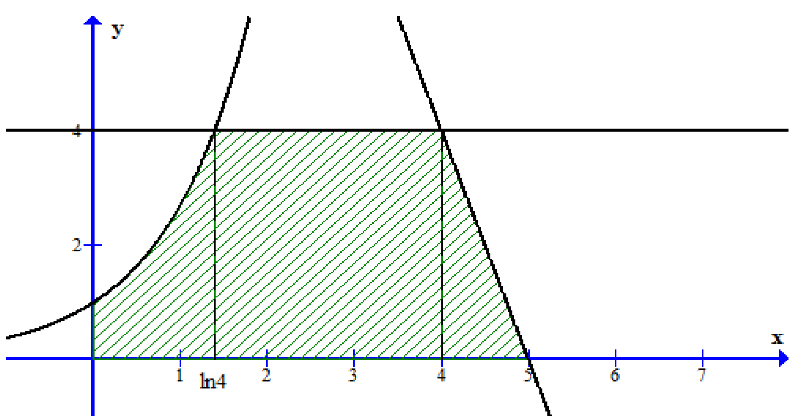
(b)

Volume	Integral expression	Value
A rotated about x-axis	$V_x = \pi\int_0^2 x^4\mathrm{d}x + \pi\int_2^{2\sqrt{2}} \left(8-x^2\right)^2\mathrm{d}x$	
A rotated about y-axis	$V_y = \pi\int_0^4 \left((8-y)-y\right)\mathrm{d}y = \pi\int_0^4 (8-2y)\mathrm{d}y$	
B rotated about x-axis	$V_x = \pi\int_0^2 \left((8-x^2)^2 - x^4\right)\mathrm{d}x$	
B rotated about y-axis	$V_y = \pi\int_0^4 y\,\mathrm{d}y + \pi\int_4^8 (8-y)\mathrm{d}y$	

5. (a) gradient of line = $\frac{r}{h}$, $y - 0 = \frac{r}{h}(x-0) \Rightarrow y = \frac{r}{h}x$

(b) $$V_x = \pi\int_0^h \left(\frac{r}{h}x\right)^2 \mathrm{d}x = \pi\left(\frac{r}{h}\right)^2 \int_0^h x^2\mathrm{d}x = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2}{h^2}\frac{h^3}{3} = \frac{1}{3}\pi r^2 h$$

6. (a)



y-intercept: (0,1), x-intercept: (5,0), points of intersection: (ln4,4), (4,4)

(b) (Notice that the second part is a rectangle and the third part is a triangle)

$$\int_0^{\ln 4} \mathrm{e}^x\mathrm{d}x + 4(4-\ln 4) + 2 = \ldots = 21 - 4\ln 4$$

(c) (Notice that the second part is a cylinder and the third part is a cone)

$$\pi\int_0^{\ln 4} \mathrm{e}^{2x}dx + \pi 4^2(4-\ln 4) + \frac{1}{3}\pi 4^2 = \ldots = \left(\frac{461}{6} - 16\ln 4\right)\pi$$

(d) $$\pi\int_0^4 \left[(5-y/4)^2 - (\ln y)^2\right]\mathrm{d}y$$

A. Exam style questions (SHORT)

7. (a) $V = \int_a^b \pi y^2\,dx = \pi\int_0^2 \left(2x - x^2\right)^2 dx$

(b) Volume $= \pi\int_0^2 \left(4x^2 - 4x^3 + x^4\right)dx = \pi\left[4\frac{x^3}{3} - 4\frac{x^4}{4} + \frac{x^5}{5}\right]_0^2 = \frac{16\pi}{15}$

OR directly by GDC: Volume = 3.35 $\left(= \frac{16\pi}{15} = 1.07\pi\right)$

8. $\int \pi y^2\,dx = \pi\int \left(x^{\frac{1}{2}}\right)^2 dx = \frac{x^2}{2}$ $\quad V = \pi\left[\frac{x^2}{2}\right]_0^a = \frac{\pi a^2}{2}$

$\frac{1}{2}\pi a^2 = 0.845\pi \iff a^2 = 1.69 \iff a = 1.3$

9. (i) $V = \pi\int_1^{1.5} f(x)^2\,dx$

(ii) $V = 105$ (or 33.3π)

10. (a) At $x = 0$ $y = e^0 = 1$ $\quad P(0, 1)$

(b) $V = \pi\int_0^{\ln 2} (e^{x/2})^2\,dx$

(c) $V = \int_0^{\ln 2} e^x dx = \pi[e^x]_0^{\ln 2} = \pi[e^{\ln 2} - e^0] = \pi[2 - 1] = \pi$

11. (a) $\int 3\sin^2 x\cos x\,dx = \sin^3 x + c$

(b) $V = \int_a^b \pi y^2 dx = \int_0^{\frac{\pi}{2}} \pi\left(\sqrt{3}\sin x\cos^{\frac{1}{2}} x\right)^2 dx = \pi\int_0^{\frac{\pi}{2}} 3\sin^2 x\cos x\,dx$

$= \pi\left[\sin^3 x\right]_0^{\frac{\pi}{2}} = \pi(1 - 0) = \pi$

12. $V = \pi\int_0^{\frac{\pi}{4}} \left(\sin 3x\right)^2 dx = 0.476\pi = 1.4954\ldots \cong 1.50$

13. (a) $V = \pi\int_1^3 \left(\frac{\ln x}{x^3}\right)^2 dx = 0.0458$

(b) $A = \int_1^3 \frac{\ln x}{x^3} dx$

But $\int \frac{\ln x}{x^3} dx = \int x^{-3}\ln x\,dx = \frac{x^{-2}}{-2}\ln x - \int \frac{x^{-2}}{-2}\frac{1}{x} dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3} dx = -\frac{\ln x}{2x^2} + \frac{1}{2}\frac{x^{-2}}{-2} + c = -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$

Hence

$A = \left[-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right]_1^3 = \left(-\frac{\ln 3}{18} - \frac{1}{36}\right) - \left(0 - \frac{1}{4}\right) = -\frac{\ln 3}{18} - \frac{1}{36} + \frac{1}{4} = \frac{4 - \ln 3}{18}$

14. $V = \pi \int_0^k e^{2x} dx = \frac{\pi}{2}[e^{2x}]_0^k = \frac{\pi}{2}(e^{2k} - 1)$

15. $V = \pi \int_0^{2a} y^2 dx = \pi \int_0^{2a} 8a(2a - x) dx = 8\pi a \int_0^{2a} (2a - x) dx$

$$= 8\pi a\left[2ax - \frac{x^2}{2}\right]_0^{2a}$$

$$= 8\pi a(4a^2 - 2a^2)$$

$$= 16\pi a^3$$

16. (a) $A = \int_0^a (ax + 2) - (x^2 + 2) dx = \int_0^a (ax - x^2) dx = \left[a\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}$ units^2

(b) $V_x = \pi \int_0^a \left((ax + 2)^2 - (x^2 + 2)^2\right) dx = \pi \int_0^a (a^2x^2 + 4ax + 4 - x^4 - 4x^2 - 4) dx$

$$= \pi\left[\frac{1}{3}a^2x^3 + 2ax^2 - \frac{1}{5}x^5 - \frac{4}{3}x^3\right]_0^a = \pi\left(\frac{2}{15}a^5 + \frac{2}{3}a^3\right) \text{ units}^3$$

(c) $V_y = \pi \int_2^{\alpha^2+2} [(y - 2) - (\frac{y-2}{a})^2] dy$

17. $V = \pi \int_1^e \left(\frac{\ln x}{x}\right)^2 dx$. Integration by parts

$u = (\ln x)^2, \frac{dv}{dx} = \frac{1}{x^2}$ $\quad$ $\frac{du}{dx} = \frac{2\ln x}{x}, v = -\frac{1}{x}$

$$\Rightarrow V = \pi\left(-\frac{(\ln x)^2}{x} + 2\int \frac{\ln x}{x^2} dx\right)$$

$u = \ln x, \frac{dv}{dx} = \frac{1}{x^2}$

$\frac{du}{dx} = \frac{1}{x}, v = -\frac{1}{x}$

$$\therefore \int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x}$$

$$\therefore V = \pi\left[-\frac{(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\right]_1^e$$

$$= 2\pi - \frac{5\pi}{e}$$

18. Curves meet at (0, 0) and (k, k)

$$\text{Area } = \int_0^k \left(k^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^2}{k}\right) dx = 12$$

$$\Rightarrow \left[k^{\frac{1}{2}}\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3k}\right]_0^k = 12$$

$$\Rightarrow \left[\frac{2}{3}k^2 - \frac{k^2}{3}\right] = 12$$

$$\Rightarrow \frac{k^2}{3} = 12$$

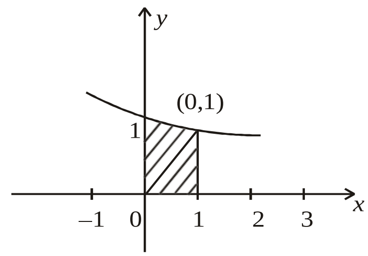
$$\Rightarrow k = 6$$

B. Exam style questions (LONG)

19. (a) $\int_0^1 e^{-kx}dx = \left[-\frac{1}{k}e^{-kx}\right]_0^1 = -\frac{1}{k}(e_{-k} - e_0) = -\frac{1}{k}(e_{-k} - 1) = -\frac{1}{k}(1 - e_{-k})$

(b) $k = 0.5$

(i)



(ii) Shading (see graph)

(iii) Area = $\int_0^1 e^{-kx}dx = \frac{1}{0.5}(1 - e^{05})$

(c) Volume = $\pi\int_0^1 e^{-2kx}]dx = \pi\left[\frac{e^{-2kx}}{-2k}\right]_0^1 = \pi\left[\frac{e^{-2k}}{-2k} - \frac{1}{-2k}\right] = \frac{\pi}{2k}(1 - e^{-2k})$

20. (a) (i) $a = 1.3302... \cong 1.33$

(ii) $p = (10x + 2) - (1 + e^{2x}) = 10x - e^{2x} + 1$

(iii) $\frac{dp}{dx} = 10 - 2e^{2x}$

$\frac{dp}{dx} = 0 \Leftrightarrow 10 - 2e^{2x} = 0 \Leftrightarrow e^{2x} = 5 \Leftrightarrow x = \frac{1n\,5}{2}$ (= 0.805)

(iv) $p_{\max} = 5\ln 5 - 5 + 1 = 5\ln 5 - 4$

(b) $A = \int_0^{1.3302}\left[(10x+2) - (1+e^{2x})\right]dx = \int_0^{1.3302}(10x - e^{2x} + 1)dx$

$A \cong 3.53$

(c) $V = \pi\int_0^{1.3302}\left[(10x+2)_2 - (1+e_{2x})_2\right]dx = 168.5369... \cong 169$

$V \cong 169$

21. (a) (i) $f'(x) = \frac{x\frac{1}{x} - \ln x \times 1}{x^2} = \frac{1 - \ln x}{x^2}$

$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)2x}{x^4} = \frac{2\ln x - 3}{x^3}$

Stationary point where $f'(x) = 0 \Leftrightarrow \ln x = 1 \Leftrightarrow x = e$

$f''(e) < 0$ so maximum.

(ii) Exact coordinates $x = e$, $y = \frac{1}{e}$

(b) Solving $f''(x) = 0 \Leftrightarrow 1n\,x = \frac{3}{2} \Leftrightarrow x = e^{\frac{3}{2}}$ (4.48)

(c) Area = $\int_1^5 \frac{\ln x}{x}\,dx$ indefinite integral by substitution or inspection: $\frac{(\ln x)^2}{2} + c$

Area = $\left[\frac{(\ln x)^2}{2}\right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2\right) = \frac{1}{2}(\ln 5)^2$ (= 1.30)

(d) $V = \int_1^5 \pi\left(\frac{\ln x}{x}\right)^2 dx = 1.38$

22. (a) $f_k(x) = x\ln x - kx \Rightarrow f_k'(x) = \ln x + 1 - k$

(b) $f_0'(x) = \ln x + 1$

$f_0(x)$ increases where $f_0'(x) > 0 \Rightarrow \ln x > -1 \Rightarrow x > \frac{1}{e}$

(c) (i) Stationary points happen where $f_k'(x) = 0 \Rightarrow \ln x = k - 1 \Rightarrow x = e^{k-1}$

(ii) x intercepts are where

$$\begin{aligned} f_k(x) &= 0 \\ \Rightarrow \quad x\ln x - kx &= 0 \\ \Rightarrow \quad x(\ln x - k) &= 0 \\ \Rightarrow \quad x = 0 \text{ or } \ln x &= k \\ \Rightarrow \quad x &= e^k \\ \Rightarrow \quad (e^k, 0) & \end{aligned}$$

(d) Area = $\int_0^{e^k} |x\ln x - kx|\,dx = \left(= \int_0^{e^k}(kx - x\ln x)\,dx\right)$

Integrate $x \ln x$ by parts.

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Area = $\left|\left[\frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{kx^2}{2}\right]_0^{e^k}\right| = \frac{e^{2k}}{4}$ **Note:** Given $x \ln x - kx = f_k(x) \approx 0$ when $x = 0$.

(e) Gradient of the tangent at $A(e^k, 0)$, m is $f_k'(e^k) = \ln e^k + 1 - k$
$= 1$

Therefore, an equation of the tangent is $y = x - e^k$.

(f) The tangent forms a right angle triangle with the coordinate axes.
The perpendicular sides are each of length e^k.
Area of the triangle = $\frac{1}{2} \times e^k \times e^k = \frac{1}{2}e^{2k}$

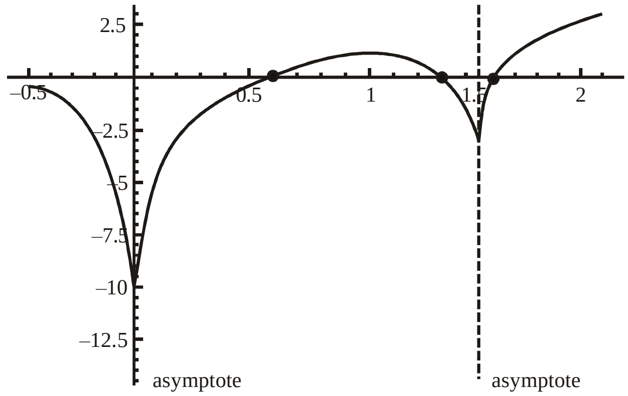
$\frac{1}{2}e^{2k} = 2\left(\frac{1}{4}e^{2k}\right)$ *i.e.* The area of the triangle is twice the area enclosed by the curve and the x-axis.

(g) Since the x-intercepts are of the form $x_k = e^k$, for $k \in \mathbb{N}$ then $x_{k+1} = e^{k+1}$

and $\frac{x_{k+1}}{x_k} = e$

Therefore, the x-intercepts $x_0, x_1, \ldots x_k, \ldots$ form a geometric sequence with $x_0 = 1$ and a common ratio of e.

23. (a) (i) $y = \ln|x^5 - 3x^2|$



[indicate the three zeros, for both asymptotes]

(ii) $f(x) = 0$ for $x = 0.599, 1.35, 1.51$

(b) $f(x)$ is undefined for$(x^5 - 3x^2) = 0$ $x^2(x^3 - 3) = 0$ Therefore, $x = 0$ or $x = 3^{1/3}$

(c) $f'(x) = \dfrac{5x^4 - 6x}{x^5 - 3x^2}\left(\text{or } \dfrac{5x^3 - 6}{x^4 - 3x}\right)$ $f'(x)$ is undefined at $x = 0$ and $x = 3^{1/3}$

(d) For the x-coordinate of the local maximum of $f(x)$, where

$0 < x < 1.5$ put $f'(x) = 0$

$5x^3 - 6 = 0$ $x = \left(\dfrac{6}{5}\right)^{\frac{1}{3}}$

(e) The required area is $A = \displaystyle\int_{0.599}^{1.35} f(x)\mathrm{d}x$

24.

(a) $\cos 2\theta = \cos(\theta + \theta)$

$\cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta \quad (= \cos^2\theta - \sin^2\theta)$

$= \cos^2\theta - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

(b) $\int \cos^2 x\,\mathrm{d}x = \dfrac{1}{2}\int(\cos 2x + 1)\mathrm{d}x$

$\int \cos 2x\,\mathrm{d}x = \dfrac{1}{2}\sin 2x$

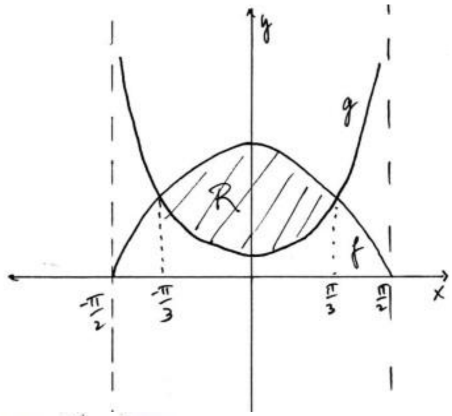
$\int \cos^2 x\,\mathrm{d}x = \dfrac{1}{4}\sin 2x + \dfrac{1}{2}x + C$

(c) Curves intersect when $f(x) = g(x)$ *i.e.* $4\cos x = \dfrac{1}{\cos x}$

$\cos^2 x = \dfrac{1}{4}\left(\cos x = \pm\dfrac{1}{2}\right)$

solving gives $x = \pm\dfrac{\pi}{3}$

(d)



(e) (i) Using $V = \int_a^b \pi y^2 \, dx$

Volume $= \pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (16\cos^2 x - \sec^2 x) \, dx$

(ii) $16 \int \cos^2 x \, dx = 16\left(\frac{1}{4}\sin 2x + \frac{1}{2}x\right)$

$\int \sec^2 x \, dx = \tan x$

Substituting limits

Volume $= 2\pi\left(4\frac{\sqrt{3}}{2} + \frac{8\pi}{3} - \sqrt{3}\right)$

$= 2\pi\left(\sqrt{3} + \frac{8\pi}{3}\right)$

25.

(a) For $x\sqrt{9-x^2}$, $-3 \le x \le 3$ and for $2\arcsin\left(\frac{x}{3}\right)$, $-3 \le x \le 3$

$\Rightarrow D$ is $-3 \le x \le 3$

(b) $V = \pi \int_0^{2.8} \left(x\sqrt{9-x^2} + 2\arcsin\frac{x}{3}\right)^2 dx$

$= 181$

(c) $\frac{dy}{dx} = (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{\frac{2}{3}}{\sqrt{1-\frac{x^2}{9}}}$

$= (9-x^2)^{\frac{1}{2}} - \frac{x^2}{(9-x^2)^{\frac{1}{2}}} + \frac{2}{(9-x^2)^{\frac{1}{2}}} \quad = \frac{9-x^2-x^2+2}{(9-x^2)^{\frac{1}{2}}} \quad = \frac{11-2x^2}{\sqrt{9-x^2}}$

(d) $\int_{-p}^{p} \frac{11-2x^2}{\sqrt{9-x^2}} dx = \left[x\sqrt{9-x^2} + 2\arcsin\frac{x}{3}\right]_{-p}^{p}$

$= p\sqrt{9-p^2} + 2\arcsin\frac{p}{3} + p\sqrt{9-p^2} + 2\arcsin\frac{p}{3}$

$= 2p\sqrt{9-p^2} + 4\arcsin\left(\frac{p}{3}\right)$

(e) $11 - 2p^2 = 0$

$p = 2.35 \quad \left(\sqrt{\dfrac{11}{2}}\right)$

(f) (i) $f''(x) = \dfrac{(9-x^2)^{\frac{1}{2}}(-4x) + x(11-2x^2)(9-x^2)^{-\frac{1}{2}}}{9-x^2}$

$= \dfrac{-4x(9-x^2) + x(11-2x^2)}{(9-x^2)^{\frac{3}{2}}}$

$= \dfrac{-36x + 4x^3 + 11x - 2x^3}{(9-x^2)^{\frac{3}{2}}}$

$= \dfrac{x(2x^2 - 25)}{(9-x^2)^{\frac{3}{2}}}$

(ii) **EITHER**

When $0 < x < 3$, $f''(x) < 0$. When $-3 < x < 0$, $f''(x) > 0$.

OR

$f''(0) = 0$

THEN

Hence $f''(x)$ changes sign through $x = 0$, giving a point of inflexion.

$x = \pm\sqrt{\dfrac{25}{2}}$ is outside the domain of f.

26. (a) $\displaystyle\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$

(b) (i) Area $= \left|\left[\dfrac{1}{3}x\sin 3x + \dfrac{1}{9}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{3\pi}{6}}\right| = \dfrac{2\pi}{9}$

(ii) Area $= \left|\left[\dfrac{1}{3}x\sin 3x + \dfrac{1}{9}\cos 3x\right]_{\frac{3\pi}{6}}^{\frac{5\pi}{6}}\right| = \dfrac{4\pi}{9}$

(iii) Area $= \left|\left[\dfrac{1}{3}x\sin 3x + \dfrac{1}{9}\cos 3x\right]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}\right| = \dfrac{6\pi}{9}$

(c) The above areas form an A.S. with $u_1 = \dfrac{2\pi}{9}$ and $d = \dfrac{2\pi}{9}$

The required area $= S_n = \dfrac{n}{2}\left[\dfrac{4\pi}{9} + \dfrac{2\pi}{9}(n-1)\right] = \dfrac{n\pi}{9}(n+1)$