[MAA 5.18-5.19] MORE INTEGRALS – FURTHER SUBSTITUTION

SOLUTIONS

MORE INTEGRALS

O. Practice questions

1.

Integral	Result
$\int 5^x \mathrm{d}x$	$\dfrac{5^x}{\ln 5} + c$
$\int \left(x^3 + 3^x \right) \mathrm{d}x$	$\dfrac{x^4}{4} + \dfrac{3^x}{\ln 3} + c$
$\int \sec^2 x \mathrm{d}x$	$\tan x + c$
$\int (\tan^2 x + 1) \mathrm{d}x \quad = \int \sec^2 x \mathrm{d}x$	$\tan x + c$
$\int \dfrac{5}{\cos^2 x} \mathrm{d}x \quad = 5\int \sec^2 x \mathrm{d}x$	$5 \tan x + c$
$\int \mathrm{cosec}^2 x \mathrm{d}x$	$-\cot x + c$
$\int (\cot^2 x + 1) \mathrm{d}x \quad = \int \mathrm{cosec}^2 x \mathrm{d}x$	$-\cot x + c$
$\int \dfrac{5}{\sin^2 x} \mathrm{d}x \quad = 5\int \mathrm{cosec}^2 x \mathrm{d}x$	$-5 \cot x + c$
$\int \sec x \tan x \mathrm{d}x$	$\sec x + c$
$\int \mathrm{cosec}\, x \cot x \mathrm{d}x$	$-\mathrm{co}\sec x + c$

2.

$\int \dfrac{7}{16 + x^2} \mathrm{d}x$	$= \dfrac{7}{4} \arctan \dfrac{x}{4} + c$
$\int \dfrac{7}{1 + 16x^2} \mathrm{d}x = \dfrac{7}{16} \int \dfrac{1}{\frac{1}{16} + x^2} \mathrm{d}x$	$= \dfrac{7}{4} \arctan 4x + c$
$\int \dfrac{7}{25 + 16x^2} \mathrm{d}x = \dfrac{7}{16} \int \dfrac{1}{\frac{25}{16} + x^2} \mathrm{d}x$	$= \dfrac{7}{20} \arctan \dfrac{4x}{5} + c$
$\int \dfrac{7}{2 + x^2} \mathrm{d}x$	$= \dfrac{7}{\sqrt{2}} \arctan \dfrac{x}{\sqrt{2}} + c$
$\int \dfrac{7}{2 + 3x^2} \mathrm{d}x = \dfrac{7}{3} \int \dfrac{1}{\frac{2}{3} + x^2} \mathrm{d}x$	$= \dfrac{7\sqrt{3}}{3\sqrt{2}} \arctan \sqrt{\dfrac{3}{2}} x + c$

3.

$\int \frac{7}{\sqrt{16-x^2}}\mathrm{d}x$	$=7\arcsin\frac{x}{4}+c$
$\int \frac{7}{\sqrt{1-16x^2}}\mathrm{d}x=\frac{7}{4}\int\frac{1}{\sqrt{\frac{1}{16}-x^2}}\mathrm{d}x$	$=\frac{7}{4}\arcsin 4x+c$
$\int \frac{7}{\sqrt{25-16x^2}}\mathrm{d}x=\frac{7}{4}\int\frac{1}{\sqrt{\frac{25}{16}-x^2}}\mathrm{d}x$	$=\frac{7}{4}\arcsin\frac{4x}{5}+c$
$\int \frac{7}{\sqrt{2-x^2}}\mathrm{d}x$	$=7\arcsin\frac{x}{\sqrt{2}}+c$
$\int \frac{7}{\sqrt{2-3x^2}}\mathrm{d}x=\frac{7}{\sqrt{3}}\int\frac{1}{\sqrt{\frac{2}{3}-x^2}}\mathrm{d}x$	$=\frac{7}{\sqrt{3}}\arcsin\sqrt{\frac{3}{2}}x+c$

4. (a) $x^2-10x+24=(x-6)(x-4)$

$$\frac{2}{x^2-10x+24}=\frac{A}{x-6}+\frac{B}{x-4}\Rightarrow 2=A(x-4)+B(x-6)$$

For $x=6$, $2A=2\Leftrightarrow A=1$

For $x=4$, $-2B=2\Leftrightarrow B=-1$

Hence, $\frac{2}{x^2-10x+24}=\frac{1}{x-6}-\frac{1}{x-4}$

(b) $x^2-10x+25=(x-5)^2$

(c) $x^2-10x+26=(x-5)^2+1$

(d) (i) $\int\frac{2}{x^2-10x+24}\mathrm{d}x=\int\frac{1}{x-6}-\frac{1}{x-4}\mathrm{d}x=\ln|x-6|-\ln|x-4|+c=\ln\left|\frac{x-6}{x-4}\right|+c$

(ii) $\int\frac{2}{x^2-10x+25}\mathrm{d}x=\int\frac{2}{(x-5)^2}\mathrm{d}x=-\frac{2}{x-5}+c$

(iii) $\int\frac{2}{x^2-10x+26}\mathrm{d}x=\int\frac{2}{(x-5)^2+1}\mathrm{d}x=2\arctan(x-5)+c$

A. Exam style questions (SHORT)

5. (a) $\int\frac{e^{3x}+2e^x+4}{3e^{2x}}\mathrm{d}x=\int(\frac{1}{3}e^{x}+\frac{2}{3}e^{-x}+\frac{4}{3}e^{-2x})\mathrm{d}x=\frac{1}{3}e^{x}-\frac{2}{3}e^{-x}-\frac{2}{3}e^{-2x}+c$

(b) $\int\frac{4^x+2^x+1}{2^x}\mathrm{d}x=\int(2^x+1+2^{-x})\mathrm{d}x=\frac{2^x}{\ln 2}+x-\frac{2^{-x}}{\ln 2}+c$

6. (i) $\int\tan^2 x\mathrm{d}x=\int(\sec^2 x-1)\mathrm{d}x=\tan x-x+c$

(ii) $\int\cot^2 x\mathrm{d}x=\int(\mathrm{cosec}^2 x-1)\mathrm{d}x=-\cot x-x+c$

7. (i) $\displaystyle\int \frac{3}{x^2+9}\,dx = \frac{3}{3}\arctan\frac{x}{3}+c = \arctan\frac{x}{3}+c$

(ii) $\displaystyle\int \frac{3}{9x^2+1}\,dx = \frac{3}{9}\int \frac{1}{x^2+\frac{1}{9}}\,dx = \arctan 3x + c$

8. (i) $\displaystyle\int \frac{3}{\sqrt{9-x^2}}dx = 3\arcsin\frac{x}{3}+c$

(ii) $\displaystyle\int \frac{3}{\sqrt{1-9x^2}}dx = \frac{3}{3}\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}dx = \arcsin 3x + c$

9. (i) $\displaystyle\int \frac{7}{(x-1)^2+4}\,dx = \frac{7}{2}\arctan\frac{x-1}{2}+c$

(ii) $\displaystyle\int \frac{7}{4(x-1)^2+1}\,dx = \frac{7}{2}\arctan(2x-2)+c$

10. (i) $\displaystyle\int \frac{7}{\sqrt{4-(x-1)^2}}dx = 7\arcsin\frac{x-1}{2}+c$

(ii) $\displaystyle\int \frac{7}{\sqrt{1-4(x-1)^2}}dx = \frac{7}{2}\arcsin(2x-2)+c$

11. (i) $\displaystyle\int \frac{a}{bx^2+c}\,dx = \frac{a}{b}\int \frac{1}{x^2+\frac{c}{b}}\,dx = \frac{a}{b}\frac{\sqrt{b}}{\sqrt{c}}\arctan\frac{\sqrt{b}}{\sqrt{c}}x+c = \frac{a}{\sqrt{bc}}\arctan\frac{\sqrt{b}}{\sqrt{c}}x+c$

(ii) $\displaystyle\int \frac{a}{\sqrt{c-bx^2}}dx = \frac{a}{\sqrt{b}}\int \frac{1}{\sqrt{\frac{c}{b}-x^2}}dx = \frac{a}{\sqrt{b}}\arcsin\frac{\sqrt{b}}{\sqrt{c}}x+c$

12. (a) (RHS to LHS)

$$\frac{3}{x}-\frac{2}{(x+2)}-\frac{5}{(x+2)^2} = \frac{3(x+2)^2-2x(x+2)-5x}{x(x+2)^2}$$

$$= \frac{3x^2+12x+12-2x^2-4x-5x}{x(x+2)^2} = \frac{x^2+3x+12}{x(x+2)^2}$$

(b) $\displaystyle\int \frac{x^2+3x+12}{x(x+2)^2}\,dx = 3\ln|x|-2\ln|x+2|+\frac{5}{x+2}+c$

13. **(**Since $\Delta>0$) $x^2+6x+8=(x+2)(x+4)$

$\dfrac{6}{x^2+6x+8}=\dfrac{A}{x+2}+\dfrac{B}{x+4} \Rightarrow 6 = A(x+4)+B(x+2)$

For $x=-2$, $6=2A \Rightarrow A=3$

For $x=-4$, $6=-2B \Rightarrow B=-3$

Hence $\displaystyle\int \frac{6}{x^2+6x+8}\,dx = \int\left(\frac{3}{x+2}-\frac{3}{x+4}\right)dx = 3\ln|x+2|-3\ln|x+4|+c$ $\left(=3\ln\left|\dfrac{x+2}{x+4}\right|+c\right)$

14. (Since $\Delta < 0$) $x^2 + 6x + 13 = (x+3)^2 + 4$

$$\int \frac{6}{x^2+6x+13}\,\mathrm{d}x = \int \frac{6}{(x+3)^2+4}\,\mathrm{d}x = \frac{6}{2}\arctan\frac{x+3}{2} + c = 3\arctan\frac{x+3}{2} + c$$

15. (Since $\Delta > 0$) $x^2 - 6x + 8 = (x-2)(x-4)$

$$\frac{x+6}{x^2-6x+8} = \frac{A}{x-4} + \frac{B}{x-2} \Rightarrow x + 6 = A(x-2) + B(x-4)$$

For $x = 4$, $10 = 2A \Rightarrow A = 5$

For $x = 2$, $8 = -2B \Rightarrow B = -4$

Hence

$$\int \frac{x+6}{x^2-6x+8}\,\mathrm{d}x = \int\left(\frac{5}{x-4} - \frac{4}{x-2}\right)\mathrm{d}x = 5\ln|x-4| - 4\ln|x-2| + c \qquad \left(= \ln\frac{|x-4|^5}{|x-2|^4} + c\right)$$

16. $-x^2 - 6x + 7 = -(x+3)^2 + 16$

$$\int \frac{7}{\sqrt{-x^2-6x+7}}\,\mathrm{d}x = \int \frac{7}{\sqrt{16-(x+3)^2}}\,\mathrm{d}x = 7\arcsin\frac{x+3}{4} + c$$

17.

$$\frac{3x^2+4x+2}{x^2-6x+8} = \frac{3(x^2-6x+8)+22x-22}{x^2-6x+8} = 3 + \frac{22x-22}{x^2-6x+8}$$

$x^2 - 6x + 8 = (x-2)(x-4)$

$$\frac{22x-22}{x^2-6x+8} = \frac{A}{x-2} + \frac{B}{x-4} \Rightarrow 22x - 22 = A(x-4) + B(x-2)$$

For $x = 2$, $22 = -2A \Rightarrow A = -11$

For $x = 4$, $66 = 2B \Rightarrow B = 33$

$$\int \frac{3x^2+4x+2}{x^2-6x+8}\,\mathrm{d}x = \int\left(3 + \frac{33}{x-4} - \frac{11}{x-2}\right)\mathrm{d}x = 3x + 33\ln|x-4| - 11\ln|x-2| + c$$

FURTHER SUBSTITUTION

O. Practice questions

18. $u = \tan x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x \Rightarrow \mathrm{d}x = \dfrac{\mathrm{d}u}{\sec^2 x}$

(i) $\displaystyle\int \tan^5 x \sec^2 x\,\mathrm{d}x = \int u^5\,\mathrm{d}u = \frac{u^6}{6} + c = \frac{\tan^6 x}{6} + c$

(ii) $\displaystyle\int \frac{\sec^2 x}{\sqrt{\tan x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{u}}\,\mathrm{d}u = 2\sqrt{u} + c = 2\sqrt{\tan x} + c$

(iii) $\displaystyle\int \frac{\sec^2 x}{e^{\tan x}}\,\mathrm{d}x = \int \frac{1}{e^u}\,\mathrm{d}u = \int e^{-u}\,\mathrm{d}u = -e^{-u} + c = -\frac{1}{e^u} + c = -\frac{1}{e^{\tan x}} + c$

(iv) it is equal to $\displaystyle\int \tan^5 x \sec^2 x\,\mathrm{d}x$ (case (i). Hence $\dfrac{\tan^6 x}{6} + c$

19.

$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx = -\ln\lvert\cos x\rvert + c$
$\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx = \ln\lvert\sin x\rvert + c$
$\int \frac{2x}{x^2+7}dx = \ln(x^2+7) + c$
$\int \frac{x}{3x^2+7}dx = \frac{1}{6}\int \frac{6x}{3x^2+7}dx = \frac{1}{6}\ln(3x^2+7)+c$
$\int \frac{x^2}{3x^3+7}dx = \frac{1}{9}\int \frac{9x^2}{3x^3+7}dx = \frac{1}{9}\ln(3x^3+7)+c$
$\int \frac{6x+5}{3x^2+5x+1}dx = \ln\lvert 3x^2+5x+1\rvert + c$
$\int \frac{e^x}{e^x+5}dx = \ln(e^x+5)+c$
$\int \frac{e^{2x}}{e^{2x}+5}dx = \frac{1}{2}\int \frac{2e^{2x}}{e^{2x}+5}dx = \frac{1}{2}\ln(e^{2x}+5)+c$

20. $u = e^x \Rightarrow \frac{du}{dx} = e^x \Rightarrow dx = \frac{du}{e^x}$

$$\int \frac{e^x}{e^{2x}+1}dx = \int \frac{e^x}{u^2+1}\frac{du}{e^x} = \int \frac{1}{u^2+1}du = \arctan u + c = \arctan e^x + c$$

21. $u = x+5 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du \qquad x = u-5$

(i) $$\int \frac{3x}{x+5}dx = \int \frac{3x}{u}du = \int \frac{3(u-5)}{u}du = \int\left(3-\frac{15}{u}\right)du = 3u - 15\ln\lvert u\rvert + c$$

$$= 3(x+5) - 15\ln\lvert x+5\rvert + c = 3x - 15\ln\lvert x+5\rvert + c'$$

(ii) $$\int \frac{x}{\sqrt{x+5}}dx = \int \frac{x}{\sqrt{u}}du = \int \frac{u-5}{\sqrt{u}}du = \int (u^{\frac{1}{2}} - 5u^{-\frac{1}{2}})du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 10u^{\frac{1}{2}} + c = \frac{2}{3}(x+5)^{\frac{3}{2}} - 10(x+5)^{\frac{1}{2}} + c$$

(iii) $$\int \frac{x^2}{x+5}dx = \int \frac{x^2}{u}du = \int \frac{(u-5)^2}{u}du = \int \frac{u^2-10u+25}{u}du = \int\left(u - 10 + \frac{25}{u}\right)du$$

$$= \frac{u^2}{2} - 10u + 25\ln\lvert u\rvert + c = \frac{(x+5)^2}{2} - 10(x+5) + 25\ln\lvert x+5\rvert + c$$

22. $u = x^2 + 5 \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}x = \dfrac{\mathrm{d}u}{2x}$ $\qquad x^2 = u - 5$

(i) $\displaystyle\int x^3(x^2+5)^5\,\mathrm{d}x = \int x^3u^5\frac{\mathrm{d}u}{2x} = \frac{1}{2}\int x^2u^5\mathrm{d}u = \frac{1}{2}\int (u-5)u^5\mathrm{d}u = \frac{1}{2}\int (u^6 - 5u^5)\mathrm{d}u$

$\displaystyle = \frac{u^7}{14} - \frac{5u^6}{12} + c = \frac{(x^2+5)^7}{14} - \frac{5(x^2+5)^6}{12} + c$

(ii) $\displaystyle\int \frac{x^3}{\sqrt{x^2+5}}\mathrm{d}x = \int \frac{x^3}{\sqrt{u}}\frac{\mathrm{d}u}{2x} = \frac{1}{2}\int \frac{x^2}{\sqrt{u}}\mathrm{d}u = \frac{1}{2}\int \frac{u-5}{\sqrt{u}}\mathrm{d}u = \frac{1}{2}\int (u^{\frac{1}{2}} - 5u^{-\frac{1}{2}})\mathrm{d}u$

$\displaystyle = \frac{1}{3}u^{\frac{3}{2}} - 5u^{\frac{1}{2}} + c = \frac{1}{3}(x^2+5)^{\frac{3}{2}} - 5(x^2+5)^{\frac{1}{2}} + c$

(iii) $\displaystyle\int \frac{x^5}{x^2+5}\mathrm{d}x = \int \frac{x^5}{u}\frac{\mathrm{d}u}{2x} = \frac{1}{2}\int \frac{x^4}{u}\mathrm{d}u = \frac{1}{2}\int \frac{(u-5)^2}{u}\mathrm{d}u = \frac{1}{2}\int \frac{u^2 - 10u + 25}{u}\mathrm{d}u$

$\displaystyle = \frac{1}{2}\int\left(u - 10 + \frac{25}{u}\right)\mathrm{d}u = \frac{u^2}{4} \quad 5u \quad \frac{25}{2}\ln|u| \quad c = \frac{25}{2}\ln(x^2+5) + \frac{(x^2+5)^2}{4} \quad 5x^2 + c$

23. $u = \sin x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos x \Rightarrow \mathrm{d}x = \dfrac{\mathrm{d}u}{\cos x}$

(i) $\displaystyle\int \cos^3 x\mathrm{d}x = \int \cos^3 x\frac{\mathrm{d}u}{\cos x} = \int \cos^2 x\mathrm{d}u = \int\left(1 - \sin^2 x\right)\mathrm{d}u = \int\left(1 - u^2\right)\mathrm{d}u$

$\displaystyle = u - \frac{u^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c$

(ii) $\displaystyle\int \cos^5 x\mathrm{d}x = \int \cos^5 x\frac{\mathrm{d}u}{\cos x} = \int \cos^4 x\mathrm{d}u = \int\left(1 - \sin^2 x\right)^2\mathrm{d}u = \int\left(1 - u^2\right)^2\mathrm{d}u$

$\displaystyle = \int\left(1 - 2u^2 + u^4\right)\mathrm{d}u = u - 2\frac{u^3}{3} + \frac{u^5}{5} + c = \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$

(iii) $\displaystyle\int \sin^2 x\cos^3 x\mathrm{d}x = \int u^2\cos^3 x\frac{\mathrm{d}u}{\cos x} = \int u^2\cos^2 x\mathrm{d}u = \int u^2(1 - \sin^2 x)\mathrm{d}u$

$\displaystyle = \int\left(u^2 - u^4\right)\mathrm{d}u = \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

24. $x = 3\tan\theta \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sec^2\theta \Rightarrow \mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$

$\displaystyle\int \frac{1}{9+x^2}\mathrm{d}x = \int \frac{1}{9+9\tan^2\theta}3\sec^2\theta\,\mathrm{d}\theta = \frac{1}{3}\int \frac{\sec^2\theta}{1+\tan^2\theta}\mathrm{d}\theta = \frac{1}{3}\int 1\mathrm{d}\theta = \frac{1}{3}\theta + c$

But $\theta = \arctan\dfrac{x}{3}$

Hence the integral equals $\dfrac{1}{3}\arctan\dfrac{x}{3} + c$

25. $x = 3\sin\theta \Rightarrow \dfrac{dx}{d\theta} = 3\cos\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$$\int \frac{1}{\sqrt{9-x^2}}dx = \int \frac{1}{\sqrt{9-9\sin^2\theta}}3\cos\theta\, d\theta = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta = \int 1 d\theta = \theta + c$$

But $\theta = \arcsin\dfrac{x}{3}$

Hence the integral equals $\arcsin\dfrac{x}{3} + c$

26. (a) $\displaystyle\int \frac{1}{x^2+2}dx = \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + c$

(b) $\displaystyle\int \frac{x}{x^2+2}dx = \frac{1}{2}\int \frac{2x}{x^2+2}dx = \frac{1}{2}\ln(x^2+2) + c$

(c) $\displaystyle\int \frac{x^2}{x^2+2}dx = \int \frac{x^2+2-2}{x^2+2}dx = \int \left(1 - \frac{2}{x^2+2}\right)dx = x - \frac{2}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + c$

(d) **METHOD A**

Let $u = x^2+2 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow dx = \dfrac{du}{2x}$

$$\int \frac{x^3}{x^2+2}dx = \int \frac{x^3}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{x^2}{u}du = \frac{1}{2}\int \frac{u-2}{u}du = \frac{1}{2}\int \left(1 - \frac{2}{u}\right)du$$

$$= \frac{1}{2}\left(u - \ln|u|\right) + c = \frac{x^2+2}{2} - \frac{1}{2}\ln(x^2+2) + c = \frac{x^2}{2} - \frac{1}{2}\ln(x^2+2) + c'$$

METHOD B (we simplify by using in fact long division on x^3 by x^2+2)

$$\int \frac{x^3}{x^2+2}dx = \int \frac{x(x^2+2)-2x}{x^2+2}dx = \int \left(x - \frac{2x}{x^2+2}\right)dx = \frac{x^2}{2} - \ln(x^2+2) + c$$

A. Exam style questions (SHORT)

27.

Let $u = \ln y \Rightarrow du = \dfrac{1}{y}dy$

$$\int \frac{\tan(\ln y)}{y}dy = \int \tan u\, du$$

$$= \int \frac{\sin u}{\cos u}du = -\ln|\cos u| + c$$

EITHER

$$\int \frac{\tan(\ln y)}{y}dy = -\ln|\cos(\ln y)| + c$$

OR

$$\int \frac{\tan(\ln y)}{y}dy = \ln|\sec(\ln y)| + c$$

28. In a similar way we obtain $\ln\left|\sin(\ln y)\right| + c$

29. $y = 2 - x \Rightarrow \frac{dy}{dx} = -1 \Rightarrow dx = -du$, $\quad x = 2 - y$

$$I = \int \left(\frac{2-y}{y}\right)^2 (-dy) = -\int \left(\frac{4}{y^2} - \frac{4}{y} + 1\right) dy = \frac{4}{y} + 4\ln|y| - y + c$$

$$= \frac{4}{2-x} + 4\ln|2-x| - (2-x) + c$$

30. Let $u = \frac{1}{2}x + 1 \Rightarrow \frac{du}{dx} = \frac{1}{2} \Rightarrow dx = 2du \qquad x = 2(u-1)$

Then $\int x\left(\frac{1}{2}x + 1\right)^{1/2} dx$

$$= \int 2(u-1) \times u^{1/2} \times 2du$$

$$= 4\int (u^{3/2} - u^{1/2})\, du$$

$$= 4\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$$

$$= 4\left[\frac{2}{5}\left(\frac{1}{2}x + 1\right)^{5/2} - \frac{2}{3}\left(\frac{1}{2}x + 1\right)^{3/2}\right] + C$$

31. Substituting $u = x + 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du \qquad x = u - 2$

$$\int \frac{x^3}{(x+2)^2}dx = \int \frac{(u-2)^3}{u^2}du = \int \frac{u^3 - 6u^2 + 12u - 8}{u^2}du$$

$$= \int u\, du + \int (-6)\, du + \int \frac{12}{u}du + \int 8u^{-2}du$$

$$= \frac{u^2}{2} - 6u + 12\ln|u| + 8u^{-1} + c$$

$$= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c$$

32. $x = \sqrt{5}\sin\theta \Rightarrow \frac{dx}{d\theta} = \sqrt{5}\cos\theta \Rightarrow dx = \sqrt{5}\cos\theta\, d\theta$

$$\int \sqrt{5 - x^2}\,dx = \int \sqrt{5 - 5\sin^2\theta}\,\sqrt{5}\cos\theta\, d\theta = 5\int \sqrt{1 - \sin^2\theta}\,\cos\theta\, d\theta = 5\int \cos^2\theta\, d\theta$$

But $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$$\frac{5}{2}\int (1 + \cos 2\theta)\, d\theta = \frac{5}{2}\theta + \frac{5\sin 2\theta}{2\cdot 2} + c = \frac{5}{2}\theta + \frac{5}{2}\sin\theta\cos\theta + c$$

The substitution $x = \sqrt{5}\sin\theta$ gives $\theta = \arcsin\frac{x}{\sqrt{5}}$ and $\sin\theta = \frac{x}{\sqrt{5}}$

Using the right angled triangle method we also obtain $\sin\theta = \frac{\sqrt{5 - x^2}}{\sqrt{5}}$

Finally, we obtain

$$\frac{5}{2}\arcsin\frac{x}{\sqrt{5}} + \frac{x\sqrt{5 - x^2}}{2} + c$$

33. $x = 4\sin^2\theta \Rightarrow \dfrac{dx}{d\theta} = 8\sin\theta\cos\theta \Rightarrow dx = 8\sin\theta\cos\theta\, d\theta$

$$I = \int\sqrt{\frac{x}{4-x}}\,dx = \int\sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,8\sin\theta\cos\theta\,d\theta = 8\int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\sin\theta\cos\theta\,d\theta$$

$$= 8\int\frac{\sin\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta = 8\int\sin^2\theta\,d\theta$$

But $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

We obtain

$$I = 4\int(1-\cos 2\theta)\,d\theta = 4\theta - 2\sin 2\theta + c = 4\theta - 4\sin\theta\cos\theta + c$$

Since $x = 4\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{x}{4} \Rightarrow \sin\theta = \dfrac{\sqrt{x}}{2}$, $\theta = \arcsin\dfrac{\sqrt{x}}{2}$

and $\cos\theta = \dfrac{\sqrt{4-x}}{2}$ (using the triangle method)

$$I = 4\arcsin\frac{\sqrt{x}}{2} - 4\frac{\sqrt{x}}{2}\frac{\sqrt{4-x}}{2} + c = 4\arcsin\frac{\sqrt{x}}{2} - \sqrt{4x-x^2} + c$$

34. (a) **RHS to LHS:**

$$\frac{1}{x-2} - \frac{x}{x^2+4} = \frac{x^2+4-x(x-2)}{(x-2)(x^2+4)} = \frac{x^2+4-x^2+2x}{(x-2)(x^2+4)} = \frac{2x+4}{(x^2+4)(x-2)}$$

(b) $$\int\frac{2x+4}{(x^2+4)(x-2)}\,dx = \int\left(\frac{1}{x-2} - \frac{x}{x^2+4}\right)dx = \ln(x-2) - \frac{1}{2}\ln(x^2+4) + c$$

35. (i) $$\int\frac{e^x+1}{e^x}\,dx = \int(1+e^{-x})\,dx = x - e^{-x} + c$$

(ii) $$\int\frac{e^x}{e^x+1}\,dx = \ln(e^x+1) + c$$

36. $u = e^x \Rightarrow \dfrac{du}{dx} = e^x \Rightarrow dx = \dfrac{du}{e^x}$

$$\int\frac{e^x}{e^{2x}+4}\,dx = \int\frac{e^x}{u^2+4}\frac{du}{e^x} = \int\frac{1}{u^2+4}\,du = \frac{1}{2}\arctan\frac{u}{2} + c = \frac{1}{2}\arctan\frac{e^x}{2} + c$$

37. $u = 3^x \Rightarrow \dfrac{du}{dx} = 3^x\ln 3 \Rightarrow dx = \dfrac{du}{3^x\ln 3}$

$$\int\frac{3^x}{9^x+9}\,dx = \int\frac{3^x}{u^2+9}\frac{du}{3^x\ln 3} = \frac{1}{\ln 3}\int\frac{1}{u^2+9}\,du = \frac{1}{3\ln 3}\arctan\frac{u}{3} + c = \frac{1}{3\ln 3}\arctan\frac{3^x}{3} + c$$

$$= \frac{1}{3\ln 3}\arctan\left(3^{x-1}\right) + c$$

38. $\int\left(\sqrt{1-4x^2}\right)dx$

Let $2x=\sin\theta \Rightarrow 2\dfrac{dx}{d\theta}=\cos\theta \quad \Rightarrow dx=\dfrac{1}{2}\cos\theta\ d\theta$

$$\Rightarrow\int\left(\sqrt{1-4x^2}\right)dx=\int\sqrt{1-\sin^2\theta}\ \frac{1}{2}\cos\theta\ d\theta$$

$$=\int\frac{1}{2}\cos^2\theta\ d\theta$$

$$=\int\frac{1}{4}(\cos 2\theta+1)\,d\theta$$

$$=\frac{1}{8}\sin 2\theta+\frac{\theta}{4}+C$$

$$=\frac{1}{4}\left[2x\sqrt{1-4x^2}+\arcsin 2x\right]+C$$

39. Let $u=\cos x \Rightarrow \dfrac{du}{dx}=-\sin x \Rightarrow dx=-\dfrac{du}{\sin x}$

$$\int\sin^3 x\,dx=\int\sin^3 x\frac{du}{-\sin x}=-\int\sin^2 x\,du=\int\left(\cos^2 x-1\right)du=\int\left(u^2-1\right)du$$

$$=\frac{u^3}{3}-u+c=\frac{\cos^3 x}{3}-\cos x+c$$

40. $u=\sin x \Rightarrow \dfrac{du}{dx}=\cos x \Rightarrow dx=\dfrac{du}{\cos x}$

$$\int\sin^3 x\cos^3 x\,dx=\int u^3\cos^3 x\frac{du}{\cos x}=\int u^3\cos^2 x\,du=\int u^3(1-\sin^2 x)du$$

$$=\int\left(u^3-u^5\right)du=\frac{u^4}{4}-\frac{u^6}{6}+c=\frac{\sin^4 x}{4}-\frac{\sin^6 x}{6}+c$$

DEFINITE INTEGRALS

A. Exam style questions (SHORT)

41. $$\int_0^2\frac{1}{4+x^2}dx=\left[\frac{1}{2}\arctan\frac{x}{2}\right]_0^2=\frac{\pi}{8}-0=\frac{\pi}{8}$$

42. $$\int_0^k\frac{1}{4+x^2}dx=\frac{\pi}{6}\Leftrightarrow\left[\frac{1}{2}\arctan\frac{x}{2}\right]_0^k=\frac{\pi}{6}\Leftrightarrow\frac{1}{2}\arctan\frac{k}{2}=\frac{\pi}{6}\Leftrightarrow\arctan\frac{k}{2}=\frac{\pi}{3}$$

$$\Leftrightarrow\frac{k}{2}=\tan\frac{\pi}{3}\Leftrightarrow\frac{k}{2}=\sqrt{3}\Leftrightarrow k=2\sqrt{3}$$

43. $\int_a^{a^2} \frac{1}{1+x^2}\,dx = 0.22 \Rightarrow \ [\arctan x]_a^{a^2} = 0.22$

$\Rightarrow \arctan a^2 - \arctan a - 0.22 = 0$

$\Rightarrow a = 2.04$ or $a = 2.62$

OR directly by GDC using SolveN

44.

Let $u = e^x$

$du = e^x\,dx$ (or equivalent)

When $x = 0$, $u = 1$ **and** when $x = \ln 3$, $u = 3$

$$\int_0^{\ln 3} \frac{e^x}{e^{2x}+9}\,dx = \int_1^3 \frac{1}{u^2+9}\,du$$

$$= \frac{1}{3}\left[\arctan\frac{u}{3}\right]_1^3$$

$$= \frac{1}{3}\left(\arctan 1 - \arctan\frac{1}{3}\right)$$

$$= \frac{\pi}{12} - \frac{1}{3}\arctan\frac{1}{3}$$

OR

$$\frac{1}{3}\arctan\frac{1}{2}$$

45. $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta\,d\theta$

$\int\sqrt{4-x^2}\,dx = \int\sqrt{4-4\sin^2\theta}\ 2\cos\theta\,d\theta \ = 4\int\sqrt{1-\sin^2\theta}\ \cos\theta\,d\theta \ = 4\int\cos^2\theta\,d\theta$

But $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$2\int(1+\cos 2\theta)\,d\theta \ = 2\theta + \sin 2\theta + c$

<u>Notice</u>: For definite integrals of this form it is preferable to change the limits instead of going back to x

x	0	2
θ	0	$\frac{\pi}{2}$

Hence

$$\int_0^2 \sqrt{4-x^2}\,dx = [2\theta + \sin 2\theta]_0^{\pi/2} = (\pi + \sin\pi) - 0 = \pi$$

46. $x = \sin^2\theta \Rightarrow \dfrac{dx}{d\theta} = 2\sin\theta\cos\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$I = \int\sqrt{\frac{x}{1-x}}\,dx = \int\sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\,2\sin\theta\cos\theta\, d\theta$$

$$= \int\frac{\sin\theta}{\cos\theta}\,2\sin\theta\cos\theta\, d\theta = 2\int\sin^2\theta\, d\theta$$

But $\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

We obtain

$$I = \int(1-\cos 2\theta)\,d\theta = \theta - \frac{\sin 2\theta}{2} + c$$

METHOD A: Change limits (much quicker!)

x	0	$\frac{1}{2}$
θ	0	$\frac{\pi}{4}$

Hence

$$\int_0^{\frac{1}{2}}\sqrt{\frac{x}{1-x}}\,dx = \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \left[\frac{\pi}{4} - \frac{1}{2}\right] - 0 = \frac{\pi}{4} - \frac{1}{2}$$

METHOD B: Going back to x.

$I = \theta - \sin\theta\cos\theta + c$,

Since $x = \sin^2\theta$, $\sin\theta = \sqrt{x}$ and $\cos\theta = \sqrt{1-x}$ (using either a triangle or trig identity)

and $\theta = \arcsin\sqrt{x}$

$$I = \theta - \sin\theta\cos\theta + c = \arcsin\sqrt{x} + \sqrt{x}\sqrt{1-x} + c = \arcsin\sqrt{x} + \sqrt{x-x^2} + c$$

Hence,

$$\int_0^{\frac{1}{2}}\sqrt{\frac{x}{1-x}}\,dx = \left[\arcsin\sqrt{x} + \sqrt{x-x^2}\right]_0^{\frac{1}{2}} = \left[\arcsin\frac{1}{\sqrt{2}} + \frac{1}{2}\right] - 0 = \frac{\pi}{4} - \frac{1}{2}$$