[MAA 5.16-5.17] CONTINUITY - DIFFERENTIABILITY - L'HÔPITAL

SOLUTIONS

LIMITS AND CONTINUITY

O. Practice questions

1.

(a)

Domain of $f: x \in [-1,8[$	Range of $f: y \in [1,4]$

(b)

$f(-1)=2$	$f(1)=4$	$f(4)=2$	$f(6)=2$	$f(8)$ not defined

(c)

$\lim_{x\to1^-} f(x)=2$	$\lim_{x\to1^+} f(x)=2$	$\lim_{x\to1} f(x)=2$
$\lim_{x\to4^-} f(x)=2$	$\lim_{x\to4^+} f(x)=2$	$\lim_{x\to4} f(x)=2$
$\lim_{x\to6^-} f(x)=4$	$\lim_{x\to6^+} f(x)=2$	$\lim_{x\to6} f(x)$ does not exist

(d)

continuity at $x=4$	YES, since $\lim_{x\to4} f(x)=2$ and $f(4)=2$
continuity at $x=1$	NO, since $\lim_{x\to1} f(x)=2$ and $f(1)=4$. $2\neq4$
continuity at $x=6$	NO, since $\lim_{x\to6} f(x)$ does not exist

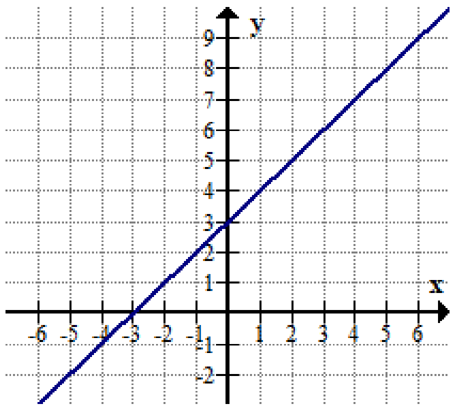
2. In fact, it is the same function as in exercise 1.

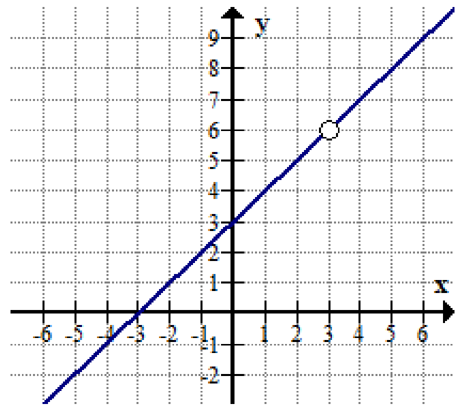
(a) (b) and (d) as in exercise 1

(b) a bit more detailed here:

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}(x^2+1)=2$
$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 2=2$
$\lim_{x\to1} f(x) = 2$
$\lim_{x\to4^-} f(x) = \lim_{x\to4^-} 2=2$
$\lim_{x\to4^+} f(x) = \lim_{x\to4^+}(x-2)=2$
$\lim_{x\to4} f(x) = 2$
$\lim_{x\to6^-} f(x) = \lim_{x\to6^-}(x-2)=4$
$\lim_{x\to6^+} f(x) = \lim_{x\to6^+} 2=2$
$\lim_{x\to6} f(x)$ does not exist

3. (a) $f(x) = x + 3$ $\qquad$ $g(x) = \dfrac{x^2 - 9}{x - 3}$





(b) $\lim_{x\to 3} f(x) = 6$ and $\lim_{x\to 3} g(x) = 6$

(c) The graphs coincide except the point at $x = 3$ where $g(x)$ is not defined.

CONTINUITY AND DIFFERENTIABILITY

O. Practice questions

4. (b) $\lim_{x\to 1^-} f(x) = 3$, $\lim_{x\to 1^+} f(x) = 4$ $\Rightarrow$ limit does not exist. Not continuous so not differentiable.

(c) at $x = 3$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 4$ so continuous

Not differentiable at $x = 3$ since the gradient changes from 0 to -2

5. (a) $a = 2$ (b) $b = -2$

(c) gradient changes from 1 to -2 at $x = 2$

6. (b) at $x = 0$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 2$

at $x = 3$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 2$

(c) gradient does not change at $x = 0$. It is 0

(d) $y = 2$

(e) gradient changes from 0 to 1 at $x = 3$.

7. (a) $c = 0$ $\quad 3a + b = 0$

(b) gradient before $x = 3$ is 2, gradient after $x = 3$ is a, so a = 2. So $b = -6$

(c) $\lim_{x\to 3} f(x) = 0$, $f'(3)$ does not exist, (at 3, not continuous $\Rightarrow$ not differentiable)

8. (a) $-2 = 1 + a + b \Rightarrow a + b = -3$

Gradient before $x = 1$: $-6x^2$, gradient after $x = 1$: $2x + a$, so $-6 = 2 + a \Rightarrow a = -8$

Thus $b = 5$

(b) $y \geq -11$

(c) $m = -6$, point $(1, -2)$. Tangent line: $y + 2 = -6(x - 1)$

DIFFERENTIATION FROM FIRST PRINCIPLES

O. Practice questions

9. For $f(x) = x^4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$

$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3)$$

$$= 4x^3$$

10. $$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^4 - 1^4}{h}$$

$$= \lim_{h\to 0} \frac{1 + 4h + 6h^2 + 4h^3 + h^4 - 1}{h}$$

$$= \lim_{h\to 0} \frac{4h + 6h^2 + 4h^3 + h^4}{h}$$

$$= \lim_{h\to 0}(4 + 6h + 4h^2 + h^3)$$

$$= 4$$

A. Exam style questions (SHORT)

11. For $f(x) = x^3 + 2x + 1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 + 2(x+h) + 1 - x^3 - 2x - 1}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 2h}{h}$$

$$= \lim_{h\to 0}(3x^2 + 3xh + h^2 + 2)$$

$$= 3x^2 + 2$$

12. For $f(x) = mx + c$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{m(x+h) + c - mx - c}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m$$

13. For $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$ (multiply up and down by $(\sqrt{x+h} + \sqrt{x})$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

14. For $f(x)=\dfrac{1}{x^2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right)$$

$$=\lim_{h\to 0}\frac{-2xh-h^2}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2}=\frac{-2x}{x^4}=\frac{-2}{x^3}$$

LIMITS AND L'HÔPITAL's RULE

O. Practice questions

15.

$\lim_{x\to 1}\dfrac{x^2+1}{(x-1)^2}=+\infty$
$\lim_{x\to 1^+}\dfrac{x^2-1}{(x-1)^2}=\lim_{x\to 1^+}\dfrac{x+1}{x-1}=+\infty$
$\lim_{x\to 1^-}\dfrac{x^2-1}{(x-1)^2}=\lim_{x\to 1^-}\dfrac{x+1}{x-1}=-\infty$

16. (a) $\lim_{x\to 0}\dfrac{e^{2x}-1}{2x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{2e^{2x}}{2}=1$

(b) $\lim_{x\to 0}\dfrac{e^{2x}-1-2x}{x^2}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{2e^{2x}-2}{2x}=\lim_{x\to 0}\dfrac{e^{2x}-1}{x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{2e^{2x}}{1}=2$

17.

$\lim_{x\to 0}\dfrac{3e^{2x}+5}{4e^{2x}+6}=\dfrac{8}{10}=\dfrac{4}{5}$	
$\lim_{x\to -\infty}\dfrac{3e^{2x}+5}{4e^{2x}+6}=\dfrac{5}{6}$	(since $e^{2x}\to 0$)
$\lim_{x\to +\infty}\dfrac{3e^{2x}+5}{4e^{2x}+6}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to +\infty}\dfrac{6e^{2x}}{8e^{2x}}=\lim_{x\to +\infty}\dfrac{6}{8}=\dfrac{3}{4}$	

18. (a)

(i) $\lim_{x\to\pm\infty}\dfrac{ax+b}{dx+e}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to\pm\infty}\dfrac{a}{d}=\dfrac{a}{d}$
(ii) $\lim_{x\to\pm\infty}\dfrac{ax^2+bx+c}{dx^2+ex+f}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to\pm\infty}\dfrac{2ax+b}{2dx+e}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to\pm\infty}\dfrac{2a}{2d}=\dfrac{a}{d}$
(iii) $\lim_{x\to\pm\infty}\dfrac{ax+b}{dx^2+ex+f}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to\pm\infty}\dfrac{a}{2dx+e}=0$

(b) They give the horizontal asymptote .

for (i) and (ii) the line $y=\dfrac{a}{d}$, for (iii) the line $y=0$

19.

$\lim\limits_{x\to+\infty}\dfrac{3x^2+x+e^x}{x^2+5x+3}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim\limits_{x\to+\infty}\dfrac{6x+1+e^x}{2x+5}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim\limits_{x\to+\infty}\dfrac{6+e^x}{2}=+\infty$	
$\lim\limits_{x\to-\infty}\dfrac{3x^2+x+e^x}{x^2+5x+3}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim\limits_{x\to-\infty}\dfrac{6x+1+e^x}{2x+5}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim\limits_{x\to-\infty}\dfrac{6+e^x}{2}=3$	(since $e^x\to 0$)

20.

(a) $\lim\limits_{x\to0}\dfrac{\sin 3x}{3x}$	$=1$ (let $y=3x$, so $y\to0$ when $x\to0$)
(b) $\lim\limits_{x\to0}\dfrac{\sin 2x}{x}$	$=2\lim\limits_{x\to0}\dfrac{\sin 2x}{2x}=2$
(c) $\lim\limits_{x\to0}\dfrac{\sin 5x}{3x}$	$=\dfrac{5}{3}\lim\limits_{x\to0}\dfrac{\sin 5x}{5x}=\dfrac{5}{3}$
(d) $\lim\limits_{x\to2}\dfrac{\sin(x-2)}{x-2}$	$=1$ (let $y=x-2$, so $y\to0$ when $x\to2$)
(e) $\lim\limits_{x\to0}\dfrac{3x+2\sin x}{x}$	$=\lim\limits_{x\to0}\left(3+2\dfrac{\sin x}{x}\right)=3+2=5$
(f) $\lim\limits_{x\to0^+}\dfrac{\sin x}{x^2}$	$=\lim\limits_{x\to0^+}\dfrac{1}{x}\dfrac{\sin x}{x}=+\infty$ $(+\infty\times1=+\infty)$
(g) $\lim\limits_{x\to0^+}\dfrac{\sin x}{\sqrt{x}}$	$=\lim\limits_{x\to0^+}\sqrt{x}\dfrac{\sin x}{x}=0\times1=0$
(h) $\lim\limits_{x\to0}\dfrac{\sin 2x}{x\cos x}$	$=\lim\limits_{x\to0}\dfrac{2\sin x\cos x}{x\cos x}=\lim\limits_{x\to0}\dfrac{2\sin x}{x}=2$
(i) $\lim\limits_{n\to+\infty}n\sin\dfrac{1}{n}$	$=\lim\limits_{n\to+\infty}\dfrac{\sin 1/n}{1/n}=1$ (let $y=1/n$, so $y\to0$ when $n\to+\infty$)
(j) $\lim\limits_{n\to+\infty}n^2\sin\dfrac{1}{n}$	$=\lim\limits_{n\to+\infty}n\dfrac{\sin 1/n}{1/n}=+\infty$ $(+\infty\times1=+\infty)$
(k) $\lim\limits_{n\to+\infty}n\sin\dfrac{1}{n^2}$	$=\lim\limits_{n\to+\infty}\dfrac{1}{n}\dfrac{\sin 1/n}{1/n^2}=0\times1=0$

A. Exam style questions (SHORT)

21. (a) $\lim\limits_{x\to0}\left(\dfrac{1}{x^2}-\dfrac{1}{x}\right)=\lim\limits_{x\to0}\left(\dfrac{1-x}{x^2}\right)=+\infty$

(b) $\lim\limits_{x\to0^+}\left(\dfrac{1}{x+1}-\dfrac{1}{x}\right)=\lim\limits_{x\to0^+}\dfrac{x-x-1}{(x+1)x}=\lim\limits_{x\to0^+}\dfrac{-1}{(x+1)x}=-\infty$

(c) $\lim\limits_{x\to0^-}\left(\dfrac{1}{x+1}-\dfrac{1}{x}\right)=\lim\limits_{x\to0^-}\dfrac{x-x-1}{(x+1)x}=\lim\limits_{x\to0^-}\dfrac{-1}{(x+1)x}=+\infty$

22. (i) $\lim\limits_{x\to1}\dfrac{3\ln x}{x-1} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to1}\dfrac{\frac{3}{x}}{1}=3$

(ii) $\lim\limits_{x\to1}\dfrac{x-1}{3\ln x} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to1}\dfrac{1}{\frac{3}{x}}=\dfrac{1}{3}$

23. $\lim\limits_{x\to0}\dfrac{2\cos x-2+5x^2}{x^2} \overset{\left(\frac{0}{0}\right)}{=} \lim\limits_{x\to0}\dfrac{-2\sin x+10x}{2x}=\lim\limits_{x\to0}\left(\dfrac{-\sin x}{x}+5\right)=-1+5=4$

24. using l'Hôpital's Rule

$$\lim_{x\to0}\frac{2\ln(1+e^x)-x-\ln 4}{x^2}=\lim_{x\to0}\frac{2e^x\div(1+e^x)-1}{2x}=\lim_{x\to0}\frac{2e^x\div(1+e^x)^2}{2}=\frac{1}{4}$$

25. **EITHER**

$$\lim_{x\to0}\left(\frac{1}{x}-\cot x\right)=\lim_{x\to0}\left(\frac{\tan x-x}{x\tan x}\right)$$

$=\lim\limits_{x\to0}\left(\dfrac{\sec^2 x-1}{x\sec^2 x+\tan x}\right)$, using l'Hopital

$$=\lim_{x\to0}\left(\frac{2\sec^2 x\tan x}{2\sec^2 x+2x\sec^2 x\tan x}\right)=0$$

OR

$$\lim_{x\to0}\left(\frac{1}{x}-\cot x\right)=\lim_{x\to0}\left(\frac{\sin x-x\cos x}{x\sin x}\right)$$

$=\lim\limits_{x\to0}\left(\dfrac{x\sin x}{\sin x+x\cos x}\right)$, using l'Hopital

$$=\lim_{x\to0}\left(\frac{\sin x+x\cos x}{2\cos x-x\sin x}\right)=0$$

26. Using l'Hopital's rule, $\lim\limits_{x\to1}\left(\dfrac{\ln x}{\sin 2\pi x}\right)=\lim\limits_{x\to1}\left(\dfrac{\frac{1}{x}}{2\pi\cos 2\pi x}\right)=\dfrac{1}{2\pi}$

27. $\lim\limits_{x\to0}\dfrac{\tan x}{x+x^2}=\lim\limits_{x\to0}\dfrac{\sec^2 x}{1+2x}=1$

28. $\lim\limits_{x\to1}\dfrac{1-x^2+2x^2\ln x}{1-\sin\frac{\pi x}{2}}=\lim\limits_{x\to1}\dfrac{-2x+2x+4x\ln x}{-\frac{\pi}{2}\cos\frac{\pi x}{2}}$

$\lim\limits_{x\to1}\dfrac{4+4\ln x}{\frac{\pi^2}{4}\sin\frac{\pi x}{2}}=\lim\limits_{x\to1}\dfrac{1-x^2+2x^2\ln x}{1-\sin\frac{\pi x}{2}}=\dfrac{4}{\frac{\pi^2}{4}}=\dfrac{16}{\pi^2}$

29. $\lim\limits_{x\to0}\dfrac{\tan x-x}{1-\cos x}\overset{\left(\frac{0}{0}\right)}{=}\lim\limits_{x\to0}\dfrac{\sec^2 x-1}{\sin x}\overset{\left(\frac{0}{0}\right)}{=}\lim\limits_{x\to0}\dfrac{2\sec x\sec x\tan x}{\cos x}=0$

30. $\lim_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{\cos x}{\sin x}\right)=\lim_{x\to 0}\dfrac{1-\cos x}{\sin x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{\sin x}{\cos x}=0$

31. $\lim_{x\to 0}\dfrac{2^x-1}{x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{2^x\ln 2}{1}=\ln 2$

32. $\lim_{x\to 0}\dfrac{(1+x^2)^{\frac{3}{2}}-1}{\ln(1+x)-x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{\frac{3}{2}(1+x^2)^{\frac{1}{2}}2x}{\frac{1}{1+x}-1}=\lim_{x\to 0}\dfrac{3(1+x^2)^{\frac{1}{2}}x}{\frac{-x}{1+x}}=\lim_{x\to 0}\left[-3(1+x^2)^{\frac{1}{2}}(1+x)\right]=-3$

33. (a) It takes values close to 2, so the limit seems to be 2.

(b) $\lim_{x\to 0}\dfrac{2\tan x-\tan 2x}{\sin 2x-2\sin x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to 0}\dfrac{2\sec^2 x-2\sec^2 2x}{2\cos 2x-2\cos x}=\lim_{x\to 0}\dfrac{\sec^2 x-\sec^2 2x}{\cos 2x-\cos x}$

$=\lim_{x\to 0}\dfrac{\frac{1}{\cos^2 x}-\frac{1}{\cos^2 2x}}{\cos 2x-\cos x}=\lim_{x\to 0}\dfrac{\frac{\cos^2 2x-\cos^2 x}{\cos^2 x\cos^2 2x}}{\cos 2x-\cos x}=\lim_{x\to 0}\dfrac{\cos 2x+\cos x}{\cos^2 x\cos^2 2x}=2$

B. Exam style questions (LONG)

34. (a) $f(x)=(2x+3)^2$

$f'(x)=2(2x+3)\times 2=8x+12$ and $f'(0)=12$

(b) $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\dfrac{(2(x+h)+3)^2-(2x+3)^2}{h}$

$=\lim_{h\to 0}\dfrac{4(x+h)^2+12(x+h)+9-4x^2-12x-9}{h}$

$=\lim_{h\to 0}\dfrac{4(x^2+2xh+h^2)+12x+12h-4x^2-12x}{h}$

$=\lim_{h\to 0}\dfrac{8xh+4h^2+12h}{h}$

$=\lim_{h\to 0}(8x+4h+12)$

$=8x+12$

(c) $f'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$=\lim_{h\to 0}\dfrac{(2h+3)^2-9}{h}$

$=\lim_{h\to 0}\dfrac{4h^2+12h+9-9}{h}$

$=\lim_{h\to 0}\dfrac{4h^2+12h}{h}=\lim_{h\to 0}(4h+12)=12$

35.

(a) r^{th} term $= \binom{n}{n-r} x^r h^{n-r} \left(= \frac{n!}{r!(n-r)!} x^r h^{n-r} \right)$

(b) $$\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = \lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h} \right)$$

$$= \lim_{h\to 0}\left(\frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \ldots + h^n - x^n}{h} \right)$$

$$= \lim_{h\to 0}\left(\frac{x^n + nx^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \ldots + h^n - x^n}{h} \right)$$

$$= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \ldots + h^{n-1} \right)$$

Note:	Accept first, second and last terms in the 3 lines above.

$$= nx^{n-1}$$

(c) $$x^n \times x^{-n} = 1$$

$$x^n \frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + x^{-n} \frac{\mathrm{d}(x^n)}{\mathrm{d}x} = 0$$

$$x^n \frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + x^{-n} \times nx^{n-1} = 0$$

$$x^n \frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + nx^{-1} = 0$$

$$\frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} = \frac{-nx^{-1}}{x^n} \left(= -nx^{-(1+n)} \right)$$

36. (a) $$\lim_{x\to 0} \frac{\ln(1+ax)}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{\frac{a}{1+ax}}{1} = a$$

(b) $$\lim_{n\to+\infty} n \ln\left(1+\frac{a}{n}\right) = \lim_{n\to+\infty} \frac{\ln\left(1+\frac{a}{n}\right)}{1/n} = a \qquad \text{(set } x = 1/n\text{)}$$

(c) $$\lim_{n\to+\infty} \left(1+\frac{a}{n}\right)^n = \lim_{n\to+\infty} \mathrm{e}^{\ln\left(1+\frac{a}{n}\right)^n} = \lim_{n\to+\infty} \mathrm{e}^{n\ln\left(1+\frac{a}{n}\right)} = \mathrm{e}^a$$

(d) $$\lim_{k\to+\infty} FV = \lim_{k\to+\infty} PV \left[\left(1+\frac{r/100}{k} \right)^k \right]^n = PV\left(\mathrm{e}^{r/100}\right)^n = PV\mathrm{e}^{rn/100}$$