

[MAA 5.11] DEFINITE INTEGRALS – AREAS

SOLUTIONS

DEFINITE INTEGRALS – PROPERTIES

O. Practice questions

1.

$\int_0^1 (2x+3)dx$	= 4
$\int_1^2 (2x+3)dx$	= 6
$\int_0^2 (2x+3)dx$	= 10
$\int_{-2}^2 (2x+3)dx$	= 12
$\int_0^1 (e^x + 2)dx$	= e+1
$\int_0^\pi (\sin x + \cos x)dx$	= 2

$\int_{1/x}^e dx$	= 7
$\int_0^1 e^{2x+3} dx$	$\frac{e^5 - e^3}{2}$
$\int_0^1 \frac{1}{x+1} dx$	= ln5
$\int_0^{10} x dx$	= 50
$\int_0^{10} 5 dx$	= 50
$\int_4^{10} dx$	= 6

2. (a) $f'(x) = \ln x$ (b) $3\ln 3 - 2$

3.

$\int_5^7 3f(x)dx$	24
$\int_7^5 f(x)dx$	-2
$\int_5^7 (f(x)+1)dx$	10
$\int_5^7 (f(x)+x)dx$	20
$\int_5^7 [f(x)-4g(x)]dx$	0
$\int_5^8 f(x)dx - \int_7^8 f(x)dx$	8
$3\int_5^{10} f(x)dx + \int_6^7 3f(x)dx$	24
$\int_0^{3.5} f(x-3)dx$	8
$\int_{2.5}^3 f(2x)dx$	4

4. (a) $\int \frac{1}{2x+3} dx = \frac{1}{2} \ln(2x+3) + C$

(b) $\int_0^3 \frac{1}{2x+3} dx = \left[\frac{1}{2} \ln(2x+3) \right]_0^3 = \frac{1}{2} \ln 9 - \frac{1}{2} \ln 3 = \frac{1}{2} \ln 3 = \ln \sqrt{3}$

Thus $P = 3$

(c) $\frac{1}{2} \ln(2m+3) - \frac{1}{2} \ln 3 = 1 \Leftrightarrow \ln(2m+3) - \ln 3 = 2$

$\Leftrightarrow \ln \frac{2m+3}{3} = 2 \Leftrightarrow \frac{2m+3}{3} = e^2 \Leftrightarrow 2m+3 = 3e^2 \Leftrightarrow m = \frac{3e^2 - 3}{2}$

A. Exam style questions (SHORT)

$$5. \int_3^k \frac{1}{x-2} dx = [\ln(x-2)]_3^k = \ln(k-2) - \ln 1$$

$$\ln(k-2) - \ln 1 = \ln 7 \Rightarrow k-2 = 7, \text{ thus } k = 9$$

$$6. \int_1^k \left(1 + \frac{1}{x^2}\right) dx = \left[x - \frac{1}{x} \right]_1^k = \frac{3}{2}$$

$$k - \frac{1}{k} = \frac{3}{2} \Rightarrow 2k^2 - 3k - 2 = 0 \Rightarrow (2k+1)(k-2) = 0$$

$$k = 2 \text{ since } k > 1$$

$$7. \int_0^a \cos^2 x \, dx = 0.740 \Rightarrow a = 1.047 \text{ (using a graphic display calculator)}$$

$$8. (a) \frac{1}{2} \times 10 = 5$$

$$(b) \int_1^3 g(x) dx + \int_1^3 4 dx = 10 + [4x]_1^3 = 10 + 8 = 18$$

$$9. (a) (i) \frac{16}{3}$$

$$(ii) \int_0^3 f(x) dx + \int_0^3 2 dx = 14$$

$$(b) \int_c^d f(x-2) dx = 8$$

$$c = 2, d = 5$$

$$10. (a) 10$$

$$(b) \int_1^3 3x^2 + f(x) dx = \int_1^3 3x^2 dx + \int_1^3 f(x) dx$$

$$\int_1^3 3x^2 dx = \left[x^3 \right]_1^3 = 27 - 1 = 26$$

$$\text{Thus } \int_1^3 3x^2 + f(x) dx = 26 + 5 = 31$$

$$11. (a) \frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x) \quad (= f'(4) + g'(4)) = 7 + 4 = 11$$

$$(b) \int_1^3 (g'(x) + 6) dx = \left[g(x) \right]_1^3 + \left[6x \right]_1^3 = (g(3) - g(1)) + (18 - 6) = (2 - 1) + 12 = 13$$

$$12. (a) \int_1^5 3f(x) dx = 3 \int_1^5 f(x) dx = 12$$

$$\text{Thus } \int_1^5 f(x) dx = 4$$

$$\text{Since } \int_5^1 f(x) dx = - \int_1^5 f(x) dx, \int_5^1 f(x) dx = -4$$

$$(b) I = \int_1^2 (x + f(x)) dx + \int_2^5 (x + f(x)) dx = \int_1^5 (x + f(x)) dx$$

$$\int_1^5 x dx + \int_1^5 f(x) dx$$

$$\int_1^5 x dx = \left[\frac{x^2}{2} \right]_1^5 = \frac{25}{2} - \frac{1}{2} = 12$$

$$I = 4 + 12 = 16$$

AREAS

O. Practice questions

13. (a) (i) $\int_0^{\frac{\pi}{2}} \cos x dx = 1$, (ii) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx = -2$ (iii) $\int_0^{\frac{3\pi}{2}} \cos x dx = -1$
 (b) (i) 1 (ii) 2 (iii) 3

14. (a) $\frac{32}{3}(\sqrt{2} - 1) \approx 4.42$
 (b) 10.7

15.

Region enclosed by	Expression for the area	Area
$f(x) = \cos(x^2)$ and $g(x) = e^x$, for $-1.5 \leq x \leq 0.5$.	$\int_{-1.11}^0 \cos x^2 - e^x dx$	0.282
$y = \sin x$ and $y = x^2 - 2x + 1.5$, for $0 \leq x \leq \pi$.	$\int_{0.6617}^{1.7010} (\sin x - (x^2 - 2x + 1.5)) dx$	0.271
$y = \ln x$ and $y = e^x - e$, for $x > 0$.	$\int_{0.233}^1 (\ln x - e^x + e) dx$	0.201
$y = \frac{2}{1+x^2}$ and $y = e^{x/3}$, for $-3 \leq x \leq 3$.	$\int_{-1.5247}^{0.74757} \left(\frac{2}{1+x^2} - e^{\frac{x}{3}} \right) dx =$	1.22
$f(x) = 4 - x^2$ and $g(x) = (x+1) \cos x$	$\int_{-1.92}^{2.72} (4 - x^2 - (x+1) \cos x) dx$	9.41
$y = e^{-x} - x + 1$ and the coordinate axes	$\int_0^{1.278} (e^{-x} - x + 1) dx$	1.18
$f : x \mapsto \frac{\sin x}{x}$, and x -axis for $\pi \leq x \leq 3\pi$	$\int_{\pi}^{3\pi} \left \frac{\sin x}{x} \right dx$	0.690
$y = x^3 - 3x^2 - 9x + 27$ and the line $y = x + 3$	$\int_{-3}^4 y_1 - y_2 dx$	101.75

A. Exam style questions (SHORT)

16. (a) $f'(x) = 3 \cos(x+2)$
 (b) $\int f(x) dx = \int (1 + 3 \sin(x+2)) dx = x - 3 \cos(x+2) + c$
 (c) (i) $1 + 3 \sin(x+2) = 0$, Required value of $a = 1.48$
 (ii) 3.06

17. (a) $\int_0^1 12x^2 (1-x) dx$

(b) $12 \int_{\frac{0}{3\pi}}^{\frac{1}{3}} (x^2 - x^3) dx = 12 \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^{\frac{1}{3}} = 12 \left(\frac{1}{27} - \frac{1}{16} \right) = 1$

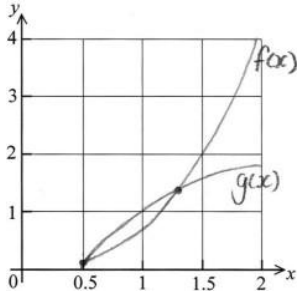
18. Area = $\int_{\frac{0}{3\pi}}^{\frac{1}{3}} \sin x dx = [-\cos x]_{\frac{0}{3\pi}}^{\frac{1}{3}} = -\cos \frac{1}{3} - (-\cos 0) = -\left(-\frac{\sqrt{2}}{2} \right) - (-1) = 1 + \frac{\sqrt{2}}{2}$

19. (a) $p = 3$

(b) $\text{Area} = \int_0^{\frac{\pi}{2}} 3 \cos x dx = [3 \sin x]_0^{\frac{\pi}{2}} = 3$ square units

20. (i) $2x + \ln(x-1) + C$
(ii) $A = \int_2^4 \left(2 + \frac{1}{x-1} \right) dx = [2x + \ln(x-1)]_2^4 = (8 + \ln 3) - (4 + \ln 1) = 4 + \ln 3$

21. (a)



(b) $A = \int_a^b g(x) - f(x) dx$ (using an appropriate definite integral)
 $a = 0.50546...$, $b = 1.227...$
 $A = 0.201$

22. (a) $x = 1$, $x = \pi$

(b) Attempting to find the area of two regions

$$B = \int_{0.5}^1 \sin x \ln x dx + \int_{\pi}^{3.5} \sin x \ln x dx$$

$$= -(0.09310... + 0.07736...)$$

$$B = 0.1704...$$

$$A = \int_1^{\pi} \sin x \ln x dx = 0.8809...$$

$$0.8809 = k \times 0.1704$$

$$k = 5.17$$

23.

$$2 + x - x^2 = 2 - 3x + x^2$$

$$\Rightarrow 2x^2 - 4x = 0$$

$$\Rightarrow 2x(x - 2) = 0$$

$$\Rightarrow x = 0, x = 2$$

$$\therefore A = \int_0^2 ((2 + x - x^2) - (2 - 3x + x^2)) dx$$

$$= \int_0^2 (4x - 2x^2) dx \text{ or equivalent}$$

$$= \left[2x^2 - \frac{2x^3}{3} \right]_0^2$$

$$= \frac{8}{3} \left(= 2\frac{2}{3} \right)$$

24. $x^2 = 2a^2 - x^2 \Leftrightarrow 2x^2 = 2a^2 \Leftrightarrow x^2 = a^2 \Leftrightarrow x = \pm a$

$$\int_{-a}^a (2a^2 - x^2 - x^2) dx = 2 \int_0^a (2a^2 - 2x^2) dx = 4 \int_0^a (a^2 - x^2) dx$$

$$= 4 \left[a^2 x - \frac{x^3}{3} \right]_0^a = 4 \left(a^3 - \frac{a^3}{3} \right) = \frac{8a^3}{3}$$

25. Area = $\int_0^k \sin 2x dx = 0.85$
 $\int_0^k \sin 2x dx = \left[-\frac{1}{2} \cos 2x \right]_0^k = -\frac{1}{2} \cos 2k + 0.5$
Equation $-\frac{1}{2} \cos 2k + 0.5 = 0.85 \quad (\Leftrightarrow \cos 2k = -0.7)$
THEN $k = 1.17$
OR directly by using SolveN : $\int_0^k \sin 2x dx = 0.85 \Rightarrow k = 1.17$
26. $\int_1^a \frac{1}{x} dx = 2 \Rightarrow [\ln x]_1^a = 2 \Rightarrow \ln a = 2 \Rightarrow a = e^2$
27. (a) 2.31
(b) (i) 1.02 (ii) 2.59
© $\int_p^q f(x) dx = 9.96$
split into two regions, make the area below the x-axis positive
28. (a) $\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x \Rightarrow \tan x = -1$
 $x = \frac{3\pi}{4}$
(b) Required area = $\int_0^{\frac{3\pi}{4}} e^x (\cos x + \sin x) dx = 7.46$ sq units
29. (a) π (3.14)
(b) (i) $\int_0^\pi e^x \sin x dx$
(ii) Area = 12.1
30. (a) (i) $f'(x) = \frac{-3(x^2-1) - (-3x)(2x)}{(x^2-1)^2} = \frac{-3x^2+3+6x^2}{(x^2-1)^2} = \frac{3x^2+3}{(x^2-1)^2} = \frac{3(x^2+1)}{(x^2-1)^2}$
(b) Area = $\int_0^a g(x) dx = \int_0^a \frac{3x^2+3}{(x^2-1)^2} dx = \left[\frac{-3x}{x^2-1} \right]_0^a = -\frac{3a}{a^2-1}$
Area = 2 $\Leftrightarrow -\frac{3a}{a^2-1} = 2 \Leftrightarrow 2a^2 + 3a - 2 = 0 \Leftrightarrow a = \frac{1}{2}$ $a = -2$ (rejected)
31. Area under parabola = $2 \int_0^a (a^2 - x^2) dx = 2 \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{4}{3} a^3$
Since $PQ = 2a$, the dimensions of the rectangle are $2a \times \frac{2}{3} a^2$.
32. (a) At A, $x = 0.753$, At B, $x = 2.45$
(b) Area $\int_{0.753}^{2.45} y dx = 1.78$
33. (a) $f(x) = 0 \Leftrightarrow a = -1.73, b = 1.73$ ($a = -\sqrt{3}, b = \sqrt{3}$)
(b) **EITHER** setting $f'(x) = 0$, **OR** directly by GDC $c = 1.15$
(c) finding 2 areas
 $-\int_{-1.73\dots}^0 f(x) dx + \int_0^{1.149\dots} f(x) dx$
area = 2.07 (accept 2.06)

B. Exam style questions (LONG)

34. (a) $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

when $x = e$, $\frac{dy}{dx} = \frac{1}{e}$

tangent line: $y - 1 = \left(\frac{1}{e} \right) (x - e) \Rightarrow y = \frac{x}{e}$

$x = 0 \Rightarrow y = \frac{0}{e} = 0$ Thud $(0, 0)$ is on line

(b) $\frac{d}{dx} (x \ln x - x) = (1) \times \ln x + x \times \left(\frac{1}{x} \right) - 1 = \ln x$

(c) Area = area of triangle - area under curve
 $= \left(\frac{1}{2} \times e \times 1 \right) - \int_1^e \ln x dx = \frac{e}{2} - [x \ln x - x]_1^e = \frac{e}{2} - \{(e \ln e - 1 \ln 1) - (e - 1)\}$
 $= \frac{e}{2} - \{e - 0 - e + 1\} = \frac{1}{2}e - 1.$

35. (a) $f(1) = 2$

$f'(x) = 4x$, $f'(1) = 4$

T: $y - 2 = 4(x - 1) \Rightarrow y = 4x - 2$

(b) $4x - 2 = 0 \Leftrightarrow x = \frac{1}{2}$

(c) **METHOD 1 (using only integrals)**

(i) Area = $\int_0^{0.5} 2x^2 dx + \int_{0.5}^1 f(x) - (4x - 2) dx$

(ii) $\int 2x^2 dx = \frac{2x^3}{3}$, $\int (4x - 2) dx = 2x^2 - 2x$

Area = $\frac{1}{12} + \frac{2}{3} - 2 + 2 - \left(\frac{1}{12} - \frac{1}{2} + 1 \right) = \frac{1}{6}$

METHOD 2 (using integral and triangle)

(i) Area = $\int_0^1 f(x) dx - \text{area of triangle} = \int_0^1 f(x) dx - \frac{1}{2}$

(ii) $\int 2x^2 dx = \frac{2x^3}{3}$

Area = $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

36. (a) $f'(x) = \frac{1}{2\sqrt{x}}$ $f'(4) = \frac{1}{4}$

gradient of normal = -4

normal; $y - 2 = -4(x - 4) \Rightarrow y = -4x + 18$

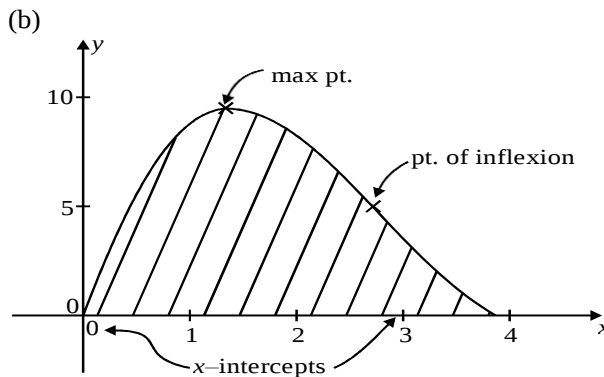
(b) $-4x + 18 = 0 \Rightarrow x = \frac{9}{4}$

(c) area of R = $\int_0^4 \sqrt{x} dx + \int_4^{4.5} (-4x + 18) dx$ OR $\int_0^4 \sqrt{x} dx + \frac{1}{2} \times 0.5 \times 2$ (triangle)

(d) $R = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^4 + 0.5 = \frac{2}{3} 4^{\frac{3}{2}} + \frac{1}{2} = \frac{16}{3} + \frac{1}{2} = \frac{35}{6}$

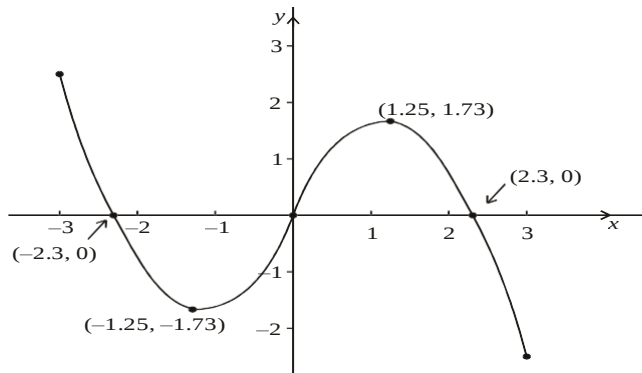
37. (a) (i) intersection points $x = 3.77, x = 8.30$
 $\int_{3.77}^{8.30} g(x) dx - \int_{3.77}^{8.30} f(x) dx = \int_{3.77}^{8.30} ((-4 \cos(0.5x) + 2) - (\ln(3x - 2) + 1)) dx$
(ii) $A = 6.46$
(b) (i) $f'(x) = \frac{3}{3x - 2}$
(ii) $g'(x) = 2 \sin(0.5x)$
(c) $f'(x) = g'(x)$
 $x = 1.43, x = 6.10$

38. (a) $y = x(x - 4)^2$
(i) $y = 0 \Leftrightarrow x = 0 \text{ or } x = 4$
(ii) $\frac{dy}{dx} = 1(x - 4)^2 + x \times 2(x - 4) = (x - 4)(x - 4 + 2x) = (x - 4)(3x - 4)$
 $\frac{dy}{dx} = 0 \Rightarrow x = 4 \text{ or } x = \frac{4}{3}$
 $x = 1 \Rightarrow \frac{dy}{dx} = (-3)(-1) = 3 > 0$
 $x = 2 \Rightarrow \frac{dy}{dx} = (-2)(2) = -4 < 0$
 $\left. \begin{array}{l} x = 1 \Rightarrow \frac{dy}{dx} = 3 > 0 \\ x = 2 \Rightarrow \frac{dy}{dx} = -4 < 0 \end{array} \right\} \Rightarrow \frac{4}{3} \text{ is a maximum}$
Note: A second derivative test may be used
 $x = \frac{4}{3} \Rightarrow y = \frac{256}{27}$ Thus $\left(\frac{4}{3}, \frac{256}{27} \right)$
(iii) $\frac{d^2y}{dx^2} = \frac{d}{dx}((x - 4)(3x - 4)) = \frac{d}{dx}(3x^2 - 16x + 16) = 6x - 16$
 $\frac{d^2y}{dx^2} = 0 \Leftrightarrow 6x - 16 = 0 \Leftrightarrow x = \frac{8}{3}$



- (c) (i) See diagram above
(ii) $0 < y < 10$ for $0 \leq x \leq 4$
So $\int_0^4 0 dx < \int_0^4 y dx < \int_0^4 10 dx \Rightarrow 0 < \int_0^4 y dx < 40$

39. (a)

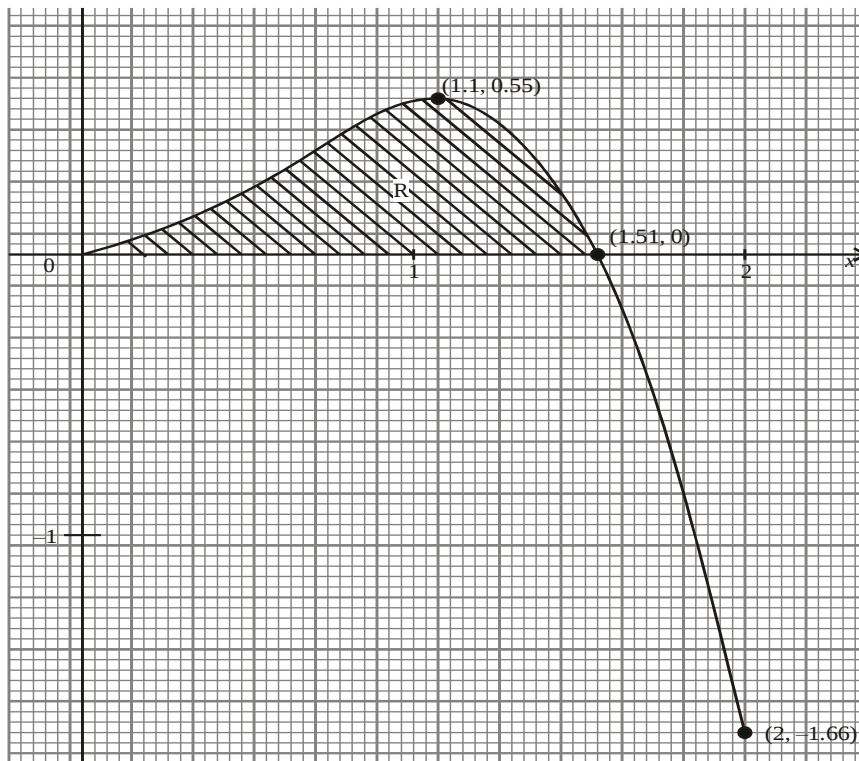


(b) $x = 2.31$

$$(c) \quad \int (\pi \sin x - x) dx = -\pi \cos x - \frac{x^2}{2} + C$$

$$\text{Required area} = \int_0^1 (\pi \sin x - x) dx = 0.944$$

40. (a)(i)&(c)(i)



(ii) Approximate positions are positive x -intercept (1.57,0), max point (1.1,0.55), end points (0,0) and (2,-1.66)

$$(b) \quad x^2 \cos x = 0 \quad x \neq 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$$

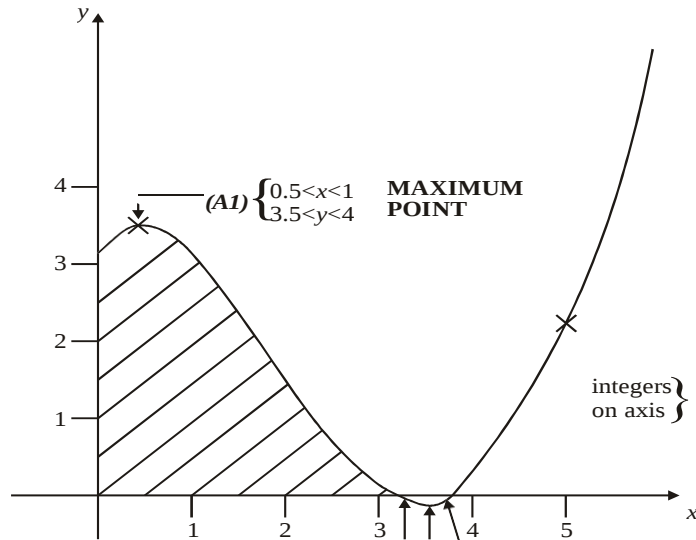
$$(c) \quad (i) \quad \text{see graph} \quad (ii) \quad \int_0^{\pi/2} x^2 \cos x \, dx$$

$$(d) \quad \text{Integral} = 0.467 \text{ by GDC OR}$$

$$\text{Integral} = \int_0^{\pi/2} x^2 \cos x + 2x \cos x - 2 \sin x \, dx = \left[\frac{\pi^2}{4} (1) + 2 \left(\frac{\pi}{2} \right) (0) - 2(1) \right] - [0 + 0 - 0]$$

$$= \frac{\pi}{2} - 2 \text{ (exact) or } 0.467 \text{ (3 s.f.)}$$

41. (a)



(b) π is a solution if and only if $\pi + \pi \cos \pi = 0$.
Now $\pi + \pi \cos \pi = \pi + \pi(-1) = 0$

(c) $x = 3.696\ 722\ 9... = 3.69672$ (6s.f.)

(d) See graph: $\int_0^\pi (\pi + x \cos x) dx$

(e) $\int_0^\pi (\pi + x \cos x) dx = 7.86960$ (6 s.f.)

42. (a) $a = -3, b = 5$

(b) (i) $f'(x) = -3x^2 + 4x + 15$
 $-3x^2 + 4x + 15 = 0 \Leftrightarrow x = -\frac{5}{3}$ or $x = 3$

(iii) $x = 3 \Rightarrow f(3) = -27 + 18 + 45 = 36$

(c) (i) At $x = 0, f'(0) = 15$
Line through $(0, 0)$ of gradient 15 $\Rightarrow y = 15x$

(ii) $-x^3 + 2x^2 + 15x = 15x \Leftrightarrow -x^3 + 2x^2 = 0 \Leftrightarrow -x^2(x - 2) = 0 \Leftrightarrow x = 2$

OR by GDC $x = 2$

(d) Area = 115 (3 s.f.)

43. (a) (i) $f'(x) = \cos x (2 \sin x \cos x) - \sin x (\sin x)^2 = \sin x \{2 \cos^2 x - \sin^2 x\}$

(ii) $f'(x) = 0 \Rightarrow \sin x \{2 \cos^2 x - \sin^2 x\} = 0 \Rightarrow \sin x \{3 \cos^2 x - 1\} = 0$

$$\Rightarrow 3 \cos^2 x - 1 = 0 \Rightarrow \cos x = \pm \sqrt{\left(\frac{1}{3}\right)}$$

$$\text{At A, } f(x) > 0, \text{ hence } \cos x = \sqrt{\left(\frac{1}{3}\right)}$$

$$(iii) f(x) = (\sin x)^2 \cos x = (1 - \cos^2 x) \cos x = \left(1 - \frac{1}{3}\right) \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} \quad (= \frac{2}{9} \sqrt{3})$$

(b) $x = \frac{\pi}{2}$

(c) (i) $\int (\cos x)(\sin x)^2 dx = \frac{1}{3} \sin^3 x + c$

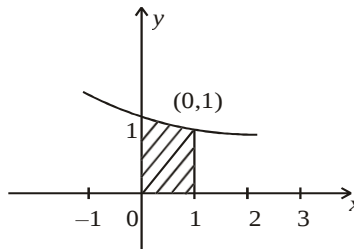
(ii) $\text{Area} = \int_0^{\pi/2} (\cos x)(\sin x)^2 dx = \frac{1}{3} \left[\left(\sin \frac{\pi}{2} \right)^3 - (\sin 0)^3 \right] = \frac{1}{3}$

(f) (i) $f'(x) = 2 \sin x \cos^2 x - \sin^3 x$
 $f'(x) = 2 \cos x \cos^2 x + 2 \sin x(-2 \cos x \sin x) - 3 \sin^2 x \cos x$
 $= 2 \cos^3 x - 4 \cos x \sin^2 x - 3 \sin^2 x \cos x$
 $= 2 \cos^3 x - 7 \cos x \sin^2 x = 2 \cos^3 x - 7 \cos x(1 - \cos^2 x)$
 $= 9(\cos x)^3 - 7 \cos x$

(ii) At C $f''(x) = 0 \Leftrightarrow 9 \cos^3 x - 7 \cos x = 0 \Leftrightarrow \cos x(9 \cos^2 x - 7) = 0$
 $\Rightarrow x = \frac{\pi}{2}$ (reject) **or** $\cos^2 x = \frac{7}{9}$
 $\cos x = \frac{\sqrt{7}}{3}$

44. (a) $\int_0^1 e^{-kx} dx = \left[-\frac{1}{k} e^{-kx} \right]_0^1 = -\frac{1}{k} (e^{-k} - e_0) = -\frac{1}{k} (e^{-k} - 1) = -\frac{1}{k} (1 - e^{-k})$

(b) $k = 0.5$
 (i)



(ii) Shading (see graph)

(iii) $\text{Area} = \int_0^1 e^{-kx} dx = \frac{1}{0.5} (1 - e^{0.5})$ **OR** $\text{Area} = 0.787$ (3 s.f.)

(c) (i) $\frac{dy}{dx} = -ke^{-kx}$

(ii) $x = 1 \quad y = 0.8 \Rightarrow 0.8 = e^{-k} \Rightarrow \ln 0.8 = -k \Rightarrow k = -\ln 0.8 (= 0.223)$

(iii) At $x = 1 \quad \frac{dy}{dx} = 0.8 \ln 0.8$ **OR** $\frac{dy}{dx} = -0.178$ (or -0.179)

45. (a) (i) $f'(x) = -\frac{3}{2}x + 1$
 $f'(2) = -\frac{2}{2}$, gradient of the normal $\left(\frac{1}{2} \right)$
 $y - 3 = \frac{1}{2}(x - 2)$ (or $y = \frac{1}{2}x + 2$)

(ii) $-\frac{3}{2}x^2 + x + 4 = \frac{1}{2}x + 2 \Leftrightarrow 3x^2 - 2x - 8 = 0 \Leftrightarrow x = -\frac{4}{3} (= -1.33)$

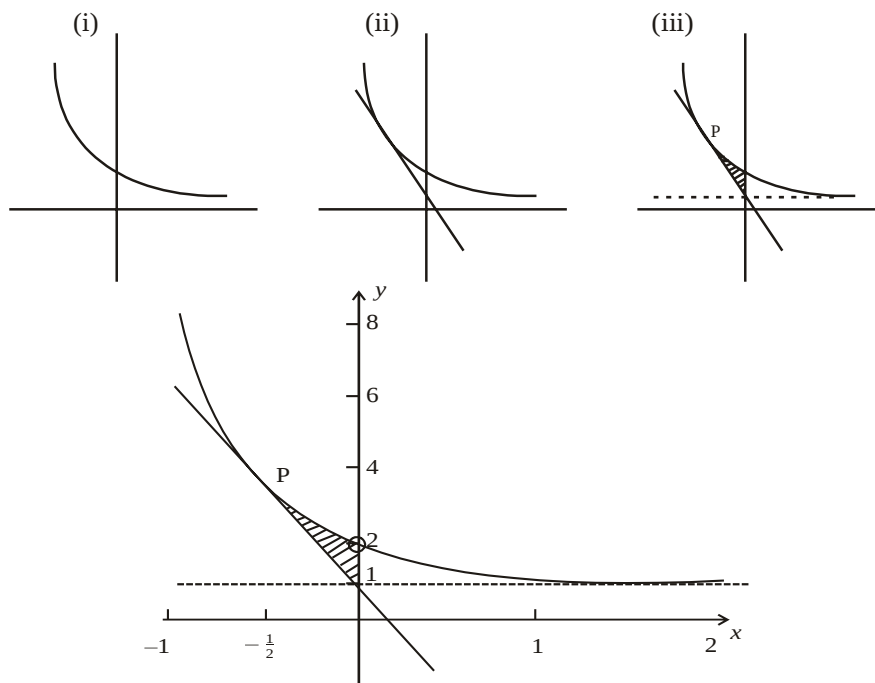
(b) (i) $\int_{-1}^2 \left(-\frac{3}{2}x^2 + x + 4 \right) dx = \left[-\frac{1}{2}x^3 + \frac{1}{2}x^2 + 4x \right]_{-1}^2$

(ii) $\text{Area} = \frac{45}{4} (= 11.25)$ (accept 11.3)

46. (a) At A, $x = 0 \Rightarrow y = \sin(e^0) = \sin(1) \Rightarrow$ coordinates of A = (0, 0.841)
 (b) $\sin(e^x) = 0 \Rightarrow e^x = \pi \Rightarrow x = \ln \pi$ (or $k = \pi$)
 (c) (i) Maximum value of sin function = 1
 (ii) $\frac{dy}{dx} = e^x \cos(e^x)$
 $\frac{dy}{dx} = 0 \Rightarrow e^x \cos(e^x) = 0 \Rightarrow e^x = 0$ (impossible) or $\cos(e^x) = 0$
 $\Rightarrow e^x = \frac{\pi}{2} \Rightarrow x = \ln \frac{\pi}{2}$
 (d) (Area = $\int_0^{\ln \pi} \sin(e^x) dx = 0.90585 = 0.906$ (3 s.f.))

47. (a) (i) $f'(x) = -2e^{-2x}$
 (ii) $f'(x)$ is always negative
 (b) (i) $y = 1 + e^{-2x - \frac{1}{2}} = 1 + e^{-\frac{1}{2}}$
 (ii) $f'\left(-\frac{1}{2}\right) = -2e^{-2(-\frac{1}{2}) - \frac{1}{2}} = -2e^{-\frac{1}{2}}$
 (c) $y - (1 + e^{-\frac{1}{2}}) = -2e^{-2x - \frac{1}{2}}$
 $y = -2ex + 1$ ($y = -5.44x + 1$)

(d)



$$(iv) \text{ Area} = \int_{-\frac{1}{2}}^0 [(1 + e^{-2x}) - (-2ex)] dx = \int_{-\frac{1}{2}}^0 [(e^{-2x} + 2ex)] dx = \left[-\frac{1}{2} e^{-2x} + ex^2 \right]_{-\frac{1}{2}}^0$$

$$= 0.1795 \dots = 0.180 \text{ (3 s.f.)}$$

OR directly by GDC Area = 0.180

48. (a) $a = 1 - \pi$ $b = 1 + \pi$

(b) (i) $\int_{-2.14}^1 h(x) dx - \int_1^2 h(x) dx$ **OR** $\int_{-2.14}^1 h(x) dx + \left| \int_1^2 h(x) dx \right|$

(ii) $5.141... - (-0.1585...) = 5.30$

(c) (i) $y = 0.973$ (ii) $-0.240 < k < 0.973$

49. (a) (i) $f'(x) = -x + 2$ (ii) $f'(0) = 2$

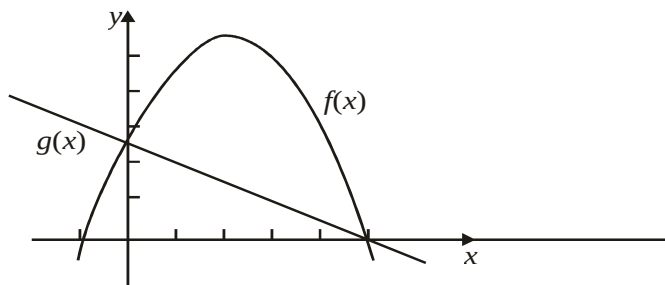
(b) y-intercept : At $x = 0, y = 2.5$

Gradient of tangent $= f'(0) = 2 \Rightarrow$ gradient of normal $= \frac{1}{2} (= -0.5)$

the normal is $y - 2.5 = -0.5(x - 0) \Leftrightarrow (y = -0.5x + 2.5)$

(c) (i) **EITHER** solving $-0.5x^2 + 2x + 2.5 = -0.5x + 2.5 \Leftrightarrow x = 0$ or $x = 5$

OR



Curves intersect at $x = 0, x = 5$

(ii) Curve and normal intersect when $x = 0$ or $x = 5$

Other point is when $x = 5 \Rightarrow y = -0.5(5) + 2.5 = 0$ (so other point $(5, 0)$)

(d) (i) $\text{Area} = \int_0^5 (f(x) - g(x)) dx = \int_0^5 (-0.5x^2 + 2x + 2.5) dx - \frac{1}{2} \times 5 \times 2.5$

(the second integral is the area of triangle)

(ii) $A_1 = \frac{50}{3}, A_2 = \frac{25}{4}$

$\text{Area} = \frac{50}{3} - \frac{25}{4} = \frac{125}{12}$ (or 10.4 (3s.f.))

50. (a) intersection points at $x = -1$ and $x = 1$

$\int_{-1}^1 e^x (1 - x^2) dx$

(b) $f(0) = 1$. Thus $P(0, 1)$

(c) $f'(0) = 1$, gradient of the normal $= -1$

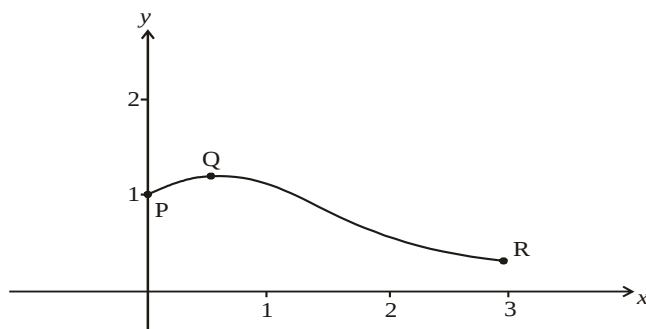
$y - 1 = -1(x - 0) \Leftrightarrow y - 1 = -x \Leftrightarrow y = -x + 1 \Leftrightarrow x + y = 1$

(d) (i) intersection points at $x = 0$ and $x = 1$

$\int_0^1 (e^x (1 - x^2) - (1 - x)) dx$ **OR** $\int_0^1 f(x) dx - \int_0^1 (1 - x) dx$

(ii) area $R = 0.5$

51. (a)



(b) (i) $f'(x) = 2e^{-x} + (2x + 1)(-e^{-x}) = (1 - 2x)e^{-x}$

(ii) At Q, $f'(x) = 0$

$x = 0.5, y = 2e^{-0.5}$ Q is $(0.5, 2e^{-0.5})$

(c) $1 \leq k < 2e^{-0.5}$

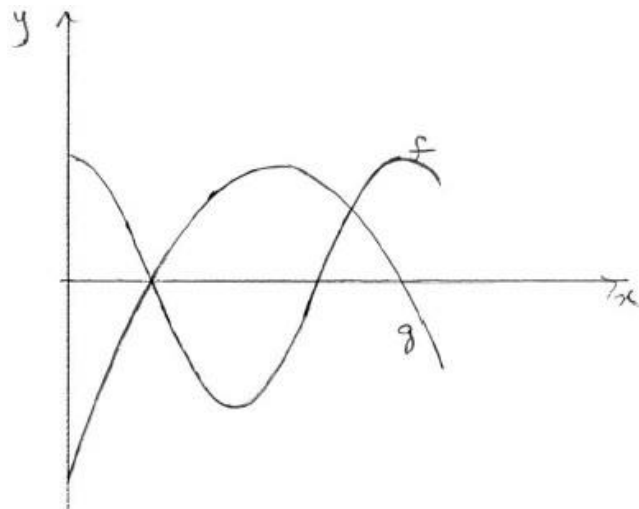
(d) At R, $y = 7e^{-3} (= 0.34850 \dots)$

Gradient of (PR) is $\frac{7e^{-3} - 1}{3} (= -0.2172)$

Equation of (PR) is $y = \left(\frac{7e^{-3} - 1}{3} \right) x + 1$ OR $y = -0.2172x + 1$

Shaded area is $\int_0^3 \left((2x+1)e^{-x} - \left(\frac{7e^{-3} - 1}{3} x + 1 \right) \right) dx = 0.529$

52. (a)



(b) (i) $(2, 0)$ (accept $x = 2$)

(ii) period = 8

(iii) amplitude = 5

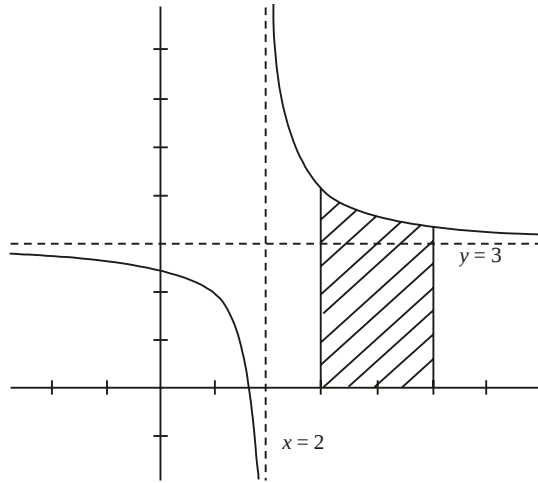
(c) (i) $(2, 0), (8, 0)$ (accept $x = 2, x = 8$)

(ii) $x = 5$ (must be an equation)

(d) intersect when $x = 2$ and $x = 6.79$

area = $\int_2^{6.79} \left(-0.5x^2 + 5x - 8 - \left(5 \cos \frac{\pi}{4} x \right) \right) dx = 27.6$

53. (a) (i)



- (ii) (Vertical asymptote) $x = 2$, (Horizontal asymptote) $y = 3$
 (b) (i) $3x + \ln(x-2) + C$
 (ii) $[3x + \ln(x-2)]_3^5 = (15 + \ln 3) - (9 + \ln 1) = 6 + \ln 3$
 (c) See graph

54. (a) limits $x = 0, x = 5$, area $= \int_0^5 f(x) dx = 52.1$

(b) area is $\int_0^a x(a-x) dx = \left[\frac{ax^2}{2} - \frac{x^3}{3} \right]_0^a = \frac{a^3}{2} - \frac{a^3}{3}$
 $\frac{a^3}{2} - \frac{a^3}{3} = 52.1 \Leftrightarrow a^3 = 6 \times 52.1 \Leftrightarrow a = 6.79$

55. (a) $f(-x) = \frac{a(-x)}{(-x)^2 + 1} = \frac{-ax}{x^2 + 1} = -f(x)$

(b) $f''(x) = 0 \Leftrightarrow 2ax(x^2 - 3) = 0 \Leftrightarrow x = 0$ or $x^2 = 3$
 $(0, 0), \left(\sqrt{3}, \frac{a\sqrt{3}}{4} \right), \left(-\sqrt{3}, -\frac{a\sqrt{3}}{4} \right)$

(c) (i) area $= \left[\frac{a}{2} \ln(x^2 + 1) \right]_2^7 = \frac{a}{2} (\ln 50 - \ln 10) = \frac{a}{2} \ln 5$

(ii) **METHOD 1**
 the shift does not change the area: $\int_3^7 f(x-1) dx = \int_2^6 f(x) dx, \frac{a}{2} \ln 5$

the factor of 2 doubles the area: $\int_4^8 2f(x-1) dx = \int_3^7 2f(x-1) dx^2 \quad \left(= 2 \int_3^7 f(x) dx \right)$

$\int_4^8 2f(x-1) dx = a \ln 5$

METHOD 2

changing variable: let $w = x - 1$, so $\frac{dw}{dx} = 1$

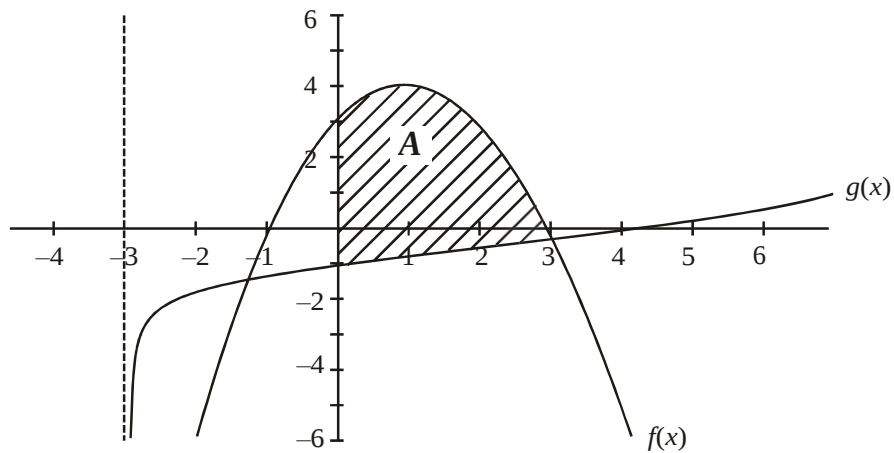
Integral $= 2 \int f(w) dw = \frac{2a}{2} \ln(w^2 + 1) + c$

Limits: when $x=4 \Rightarrow w=3$, when $x=8 \Rightarrow w=7$,

$\int_4^8 2f(x-1) dx = [a \ln(w^2 + 1)]_3^7 = a \ln 50 - a \ln 10 = a \ln 5$

56. (a) (i) $\text{gradient} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{f(a) - 0}{a - \frac{2}{3}} = \frac{a^3 - 0}{a - \frac{2}{3}} = \frac{a^3}{a - \frac{2}{3}}$
- (ii) $f'(x) = 3x^2, f'(a) = 3a^2$
- (iii) $3a^2 = \frac{a^3}{a - \frac{2}{3}} \Leftrightarrow 3a^2 \left(a - \frac{2}{3} \right) = a^3 \Leftrightarrow 3a^3 - 2a^2 = a^3 \Leftrightarrow 2a^3 - 2a^2 = 0$
- $$\Leftrightarrow 2a^2(a - 1) = 0 \Leftrightarrow a = 1$$
- (b) $\text{Area} = \int_{-2}^k (x^3 - 3x + 2) dx = \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x \right]_{-2}^k$
- $$\left(\frac{1}{4}k^4 - \frac{3}{2}k^2 + 2k \right) - (4 - 6 - 4) = \frac{1}{4}k^4 - \frac{3}{2}k^2 + 2k + 6$$
- $$\frac{1}{4}k^4 - \frac{3}{2}k^2 + 2k + 6 = 2k + 4$$
- $$\frac{1}{4}k^4 - \frac{3}{2}k^2 + 2 = 0$$
- $$k^4 - 6k^2 + 8 = 0$$
57. (a) (i) $\sin x = 0 \Leftrightarrow x = 0, x = \pi$
- (ii) $\sin x = -1 \Leftrightarrow x = \frac{3\pi}{2}$
- (b) $\frac{3\pi}{2}$
- (c) $k = \int_0^{\frac{3\pi}{2}} (6 + 6 \sin x) dx = [6x - 6 \cos x]_0^{\frac{3\pi}{2}} = 6 \left(\frac{3\pi}{2} \right) - 6 \cos \left(\frac{3\pi}{2} \right) - (-6 \cos 0)$
- $$k = 9\pi + 6$$
- (d) translation of $\left(\frac{\pi}{2}, 0 \right)$
- (e) the area under g is the same as the shaded region in f
- $$p = \frac{\pi}{2}, p = 0$$
58. (a) $13 = Ae^0 + 3 \Leftrightarrow 13 = A + 3 \Leftrightarrow A = 10$
- (b) $f(15) = 3.49 \Leftrightarrow 3.49 = 10e^{15k} + 3 \Leftrightarrow k = -0.201 \left(\text{accept } \frac{\ln 0.049}{15} \right)$
- (c) (i) $f(x) = 10e^{-0.201x} + 3$
- $$f'(x) = 10e^{-0.201x} \times -0.201 = -2.01e^{-0.201x}$$
- (ii) $f'(x) < 0$, derivative always negative
- (iii) $y = 3$
- (d) finding limits 3.8953..., 8.6940...
- $$\text{Area} = \int_{3.90}^{8.69} g(x) - f(x) dx, \int_{3.90}^{8.69} [(-x^2 + 12x - 24) - (10e^{-0.201x} + 3)] dx = 19.5$$

59. (a)



(b) (i) $x = -3$ is the vertical asymptote.

(ii) x -intercept: $x = 4.39$ ($= e^2 - 3$)

y -intercept: $y = -0.901$ ($= \ln 3 - 2$)

(c) $f(x) = g(x)$

$x = -1.34$ or $x = 3.05$

(d) (i) See graph

(ii) Area of $A = \int_0^{3.05} (4 - (1 - x)^2) - (\ln(x + 3) - 2) dx$

(iii) Area of $A = 10.6$

(e) $y = f(x) - g(x)$

$y = 5 + 2x - x^2 - \ln(x + 3)$

$$\frac{dy}{dx} = 2 - 2x - \frac{1}{x + 3}$$

Maximum occurs when $\frac{dy}{dx} = 0$

$$2 - 2x = \frac{1}{x + 3}$$

$$5 - 4x - 2x^2 = 0$$

$$x = 0.871$$

$$y = 4.63$$

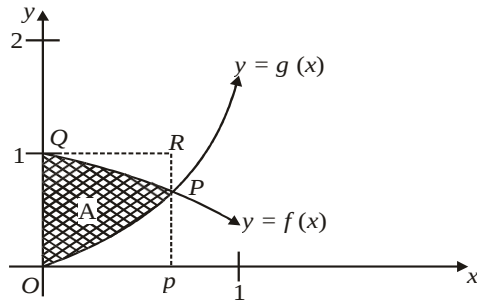
OR

Vertical distance is the difference $f(x) - g(x)$.

Maximum of $f(x) - g(x)$ occurs at $x = 0.871$.

The maximum value is 4.63.

60. (a)



(b) area $\triangle OPQ < \text{area of region A} < \text{area of rectangle OSRQ}$

$$\frac{1}{2}(1)(p) < \text{area of region A} < (p)(1)$$

$$\frac{p}{2} < \text{area of region A} < p$$

(c) Solving the equation $e^{-p^2} - e^{p^2} + 1 = 0$ using a calculator gives $p = 0.6937$ (4 decimal places)

OR the value of p may be found as follows:

$$e^{-p^2} = e^{p^2} - 1 \Rightarrow e^{2p^2} - e^{p^2} - 1 = 0$$

$$\Rightarrow e^{p^2} = \frac{1 \pm \sqrt{1+4}}{2} \Rightarrow e^{p^2} = \frac{1 \pm \sqrt{5}}{2} \text{ since } e^{p^2} > 0$$

$$\Rightarrow \text{This gives } p = \sqrt{\ln\left(\frac{1 + \sqrt{5}}{2}\right)} \approx 0.6937 \text{ (4 decimal places)}$$

(d) Area of region A = $\int_0^p (e^{-x^2} - e^{x^2} + 1) dx = 0.467$ (using a GDC)