

[MAA 2.4-2.5] COMPOSITION – INVERSE FUNCTION

SOLUTIONS

O. Practice questions

1. (a) $(f \circ g)(x) = 10 - 10x$, $(g \circ f)(x) = 50 - 10x$
 (b) $f^{-1}(x) = \frac{10-x}{2}$
 (c) $g^{-1}(10) = 2$
 (d) $(f^{-1} \circ g)(x) = \frac{10-5x}{2}$, $(g \circ f)^{-1}(x) = \frac{50-x}{10}$
 (e) $(f \circ f)(x) = 4x - 10$ $(g \circ g)(x) = 25x$

2.

Original function	Inverse function
$f(x) = x + 5$	$f^{-1}(x) = x - 5$
$f(x) = x - 5$	$f^{-1}(x) = x + 5$
$f(x) = x + 100$	$f^{-1}(x) = x - 100$
$f(x) = 3x$	$f^{-1}(x) = x / 3$
$f(x) = x / 5$	$f^{-1}(x) = 5x$
$f(x) = x^3$	$f^{-1}(x) = \sqrt[3]{x}$
$f(x) = 3x + 100$	$f^{-1}(x) = \frac{x-100}{3}$

3. (a) (i) $f(1) = 3$ (ii) $f^{-1}(1) = 5$
 (b) $x = 6$
 (c) $x = 4$

4. (a) $g(3) = 1$ $f^{-1}(3) = 4$
 (b) $(f \circ g)(2) = -1$
 (c) $(g \circ g)(3) = 5$
 (d) $x = 1$

5. $\frac{x}{x+5} = y \Leftrightarrow x = xy + 5y \Leftrightarrow x(1-y) = 5y \Leftrightarrow x = \frac{5y}{1-y}$
 $f^{-1}(x) = \frac{5x}{1-x}$ (or $\frac{-5x}{x-1}$)

6. **METHOD A**

- (a) $(g \circ f)(x) = \frac{2^x}{2^x - 2}$. Hence $(g \circ f)(3) = \frac{2^3}{2^3 - 2} = \frac{8}{6} = \frac{4}{3}$
- (b) $\frac{x}{x-2} = y \Rightarrow y(x-2) = x \Rightarrow yx - 2y = x \Rightarrow x(y-1) = 2y \Rightarrow x = \frac{2y}{y-1}$

$$g^{-1}(x) = \frac{2x}{x-1}$$

$$\text{Hence } g^{-1}(5) = \frac{10}{(5-1)} = 2.5$$

METHOD B

$$(a) \quad f(3) = 8, \quad (g \circ f)(3) = g(8) = \frac{8}{6} = \frac{4}{3}$$

$$(b) \quad \frac{x}{x-2} = 5 \Leftrightarrow x = 5x - 10 \Leftrightarrow 4x = 10 \Leftrightarrow x = \frac{5}{2}$$

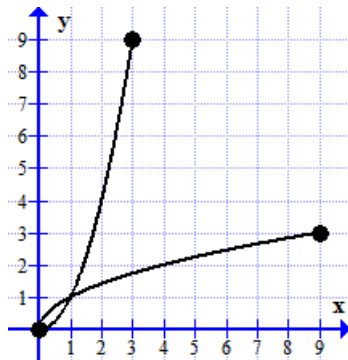
$$\text{Hence } g^{-1}(5) = \frac{5}{2} = 2.5$$

7. (a) *parabola between* $x=0$ *and* $x=3$

$$(b) \quad 0 \leq x \leq 3 \quad 0 \leq y \leq 9$$

$$(c) \quad y = \sqrt{x}$$

(d)



graph symmetric about the line $y = x$

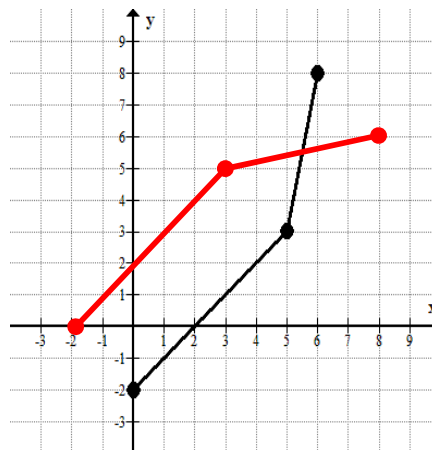
$$(e) \quad 0 \leq x \leq 9 \quad 0 \leq y \leq 3$$

8. (a) (i) $f(0) = -2$ (ii) $f(2) = 0$ (iii) $f(4) = 2$

$$(b) \quad (i) \quad f^{-1}(3) = 5 \quad (ii) \quad f^{-1}(8) = 6 \quad (iii) \quad f^{-1}(-1) = 1 \quad (iv) \quad f^{-1}(0) = 2$$

$$(c) \quad x = 2$$

$$(d) \quad x = 4$$



$$9. \quad (a) \quad \frac{2x-3}{3x-2} = y \Leftrightarrow 3xy - 2y = 2x - 3 \Leftrightarrow 3xy - 2x = 2y - 3$$

$$\Leftrightarrow x(3y - 2) = 2y - 3 \Leftrightarrow x = \frac{2y - 3}{3y - 2}$$

Hence $f^{-1}(x) = \frac{2x-3}{3x-2} = f(x)$

$$(b) \quad (f \circ f)(x) = \frac{2 \frac{2x-3}{3x-2} - 3}{3 \frac{2x-3}{3x-2} - 2} = \frac{2(2x-3) - 3(3x-2)}{3(2x-3) - 2(3x-2)} = \frac{-5x}{-5} = x$$

10. (a) $f^{-1}(x) = \sqrt[3]{x}$

(b) $f(g(x)) = x + 1 \Rightarrow [g(x)]^3 = x + 1$

so $g(x) = \sqrt[3]{x+1}$

OR $g = f^{-1} \circ (f \circ g)$, so $g(x) = \sqrt[3]{x+1}$

(b) $g(f(x)) = x + 1 \Rightarrow g(x^3) = x + 1$

so $g(x) = \sqrt[3]{x} + 1$

OR $g = (g \circ f) \circ f^{-1}$, so $g(x) = \sqrt[3]{x} + 1$

11. (a) $f = h \circ g^{-1}$

(b) $g = f^{-1} \circ h$

(c) $g = f^{-1} \circ k \circ h^{-1}$

A. Exam style questions (SHORT)

12. (a) $(h \circ g)(x) = \frac{5(3x-2)}{(3x-2)-4} = \frac{15x-10}{3x-6}$

(b) numerator = 0 $\Rightarrow$ M1) $x = \frac{2}{3}$ (= 0.667)

13. (a) $(f \circ g): x \mapsto 3(x+2) (= 3x+6)$

(b) **METHOD 1**

$$f^{-1}(x) = \frac{x}{3} \quad g^{-1}(x) = x - 2$$

$$f^{-1}(18) = \frac{18}{3} = 6 \quad g^{-1}(18) = 18 - 2 = 16$$

$$f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22$$

METHOD 2

$$3x = 18, x + 2 = 18$$

$$x = 6, x = 16$$

$$f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22$$

14. (a) $y = 2x + 1 \Rightarrow x = \frac{y-1}{2}, \quad f^{-1}(x) = \frac{x-1}{2}$

(b) $g(f(-2)) = g(-3) = 3(-3)^2 - 4 = 23$

$$(c) \quad f(g(x)) = f(3x^2 - 4) = 2(3x^2 - 4) + 1 = 6x^2 - 7$$

$$15. \quad \sqrt{3-2x} = 5 \Leftrightarrow 3-2x = 25 \Leftrightarrow -2x = 22 \Leftrightarrow x = -11$$

OR

$$\text{Let } y = \sqrt{3-2x} \Rightarrow y^2 = 3-2x \Rightarrow x = \frac{3-y^2}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{3-x^2}{2} \Rightarrow f^{-1}(5) = \frac{3-25}{2} = -11$$

16. (a) **METHOD 1**

$$f(3) = \sqrt{7} \quad (g \circ f)(3) = 7$$

METHOD 2

$$(g \circ f)(x) = \sqrt{x+4}^2 = x+4$$

$$(g \circ f)(3) = 7$$

$$(b) \quad y = \sqrt{x+4} \Rightarrow y^2 = x+4 \Rightarrow x = y^2 - 4$$

$$f^{-1}(x) = x^2 - 4$$

$$(c) \quad x \geq 0$$

17. (a) **METHOD 1**

$$f(-2) = -12$$

$$(g \circ f)(-2) = g(-12) = -24$$

METHOD 2

$$(g \circ f)(x) = 2x^3 - 8$$

$$(g \circ f)(-2) = -24$$

$$(b) \quad y = x^3 - 4 \Rightarrow y + 4 = x^3 \Rightarrow x = \sqrt[3]{y+4}$$

$$f^{-1}(x) = \sqrt[3]{x+4}$$

18. (a) **METHOD 1**

$$(f \circ g)(4) = f(g(4)) = f(1) = 2$$

METHOD 2

$$(f \circ g)(x) = \frac{2}{x-3} \quad \text{so } (f \circ g)(4) = 2$$

$$(b) \quad \text{Let } y = \frac{1}{x-3} \Rightarrow y(x-3) = 1 \Rightarrow x-3 = \frac{1}{y} \Rightarrow x = \frac{1}{y} + 3 \left(= \frac{1+3y}{y} \right)$$

$$g^{-1}(x) = \frac{1}{x} + 3 \left(= \frac{1+3x}{x} \right)$$

$$(c) \quad x \neq 0 \quad (\text{or } \mathbb{R} \setminus \{0\})$$

19. (a) For $f^{-1}(2)$, $3x + 5 = 2 \Rightarrow x = -1$

$$f^{-1}(2) = -1$$

$$(b) \quad g(f(-4)) = g(-12 + 5) = g(-7) = 2(1 + 7) = 16$$

20. (a) $(g \circ f)(x) = 2\cos x + 1$
- (b) $(f \circ g)(x) = 2\cos(2x + 1)$
- (c) $(g^{-1} \circ g)(x) = x, (g \circ g^{-1})(x) = x$
21. (a) $y = \frac{6-x}{2} \Rightarrow x = 6 - 2y \Rightarrow g^{-1}(x) = 6 - 2x$
- (b) $(f \circ g^{-1})(x) = 4[(6 - 2x) - 1] = 4(5 - 2x) = 20 - 8x$
 $20 - 8x = 4 \Leftrightarrow 8x = 16 \Leftrightarrow x = 2$
22. (a) $y = 2x - 3 \Rightarrow 2x = y + 3 \Rightarrow x = \frac{y+3}{2} \quad g^{-1}(x) = \frac{x+3}{2}$
- (b) **METHOD 1**
 $g(4) = 5 \Rightarrow f(5) = 25$
- METHOD 2**
 $f \circ g(x) = (2x - 3)^2$
 $f \circ g(4) = (2 \times 4 - 3)^2 = 25$
23. (a) $(g \circ f)(x) = 7 - 2x + 3 = 10 - 2x$
- (b) $g^{-1}(x) = x - 3$
- (c) **METHOD 1**
 $g^{-1}(5) = 2$
 $f(2) = 3$
- METHOD 2**
 $(f \circ g^{-1})(x) = 7 - 2(x - 3) = 13 - 2x$
 $(f \circ g^{-1})(5) = 3$
24. (a) $h(x) = f(2x - 5) = 6x - 15$
- (b) $6x - 15 = y \Rightarrow 6x = y + 15 \Rightarrow x = \frac{y+15}{6}$
 $h^{-1}(x) = \frac{x+15}{6}$
25. (a) $f^{-1}(x) = x^2$
 $(f^{-1} \circ g)(x) = f^{-1}(2^x) = 2^{2x}$
- (b) $2^{2x} = 16 \Rightarrow 2x = 4 \Rightarrow x = 2$
26. $f^{-1}(x) = \frac{x+5}{3} \quad g^{-1}(x) = x + 2$
 $(f^{-1} \circ g)(x) = \frac{x+3}{3} \quad (g^{-1} \circ f)(x) = 3x - 3 \Leftrightarrow$
 $\frac{x+3}{3} = 3x - 3 \Leftrightarrow x + 3 = 9x - 9 \Leftrightarrow x = \frac{12}{8} = \frac{3}{2}$

$$27. \quad (a) \quad h(x) = g(f(x)) = \frac{4}{x+2} - 1$$

$$(b) \quad y = \frac{4}{x+2} - 1 \Rightarrow y+1 = \frac{4}{x+2} \Rightarrow x+2 = \frac{4}{y+1} \Rightarrow x = \frac{4}{y+1} - 2$$

$$\text{Hence } h^{-1}(x) = \frac{4}{x+1} - 2 \quad (\text{or } \frac{2-2x}{x+1})$$

$$28. \quad (f \circ g) : x \mapsto x^3 + 1$$

$$(f \circ g)^{-1} : x \mapsto (x-1)^{1/3}$$

$$29. \quad (a) \quad g(x) = (f \circ f)(x) = \frac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \frac{x}{2x+1}$$

$$(b) \quad (g \circ g)(2) = g\left(\frac{2}{5}\right) = \frac{2}{9}$$

$$30. \quad (a) \quad y = \frac{3x-4}{x+2}$$

$$xy + 2y = 3x - 4$$

$$\text{simplifying } x(y-3) = -2y-4$$

$$\text{expressing } y \text{ in terms of } x, \quad x = \frac{2y+4}{3-y}$$

$$\text{interchanging } x \text{ and } y, \quad y = \frac{2x+4}{3-x}$$

$$f^{-1}(x) = \frac{2x+4}{3-x}$$

$$(b) \quad \text{Domain } x \neq 3$$

$$31. \quad (a) \quad f(x) = (x-3)^2 + 4 = x^2 - 6x + 9 + 4 = x^2 - 6x + 13$$

$$(b) \quad y = (x-3)^2 + 4$$

$$y-4 = (x-3)^2$$

$$\sqrt{y-4} = x-3$$

$$\sqrt{y-4} + 3 = x$$

$$\Rightarrow f^{-1}(x) = \sqrt{x-4} + 3$$

$$(c) \quad x \geq 3 \text{ and } y \geq 4$$

$$(d) \quad x \geq 4 \text{ and } y \geq 3$$

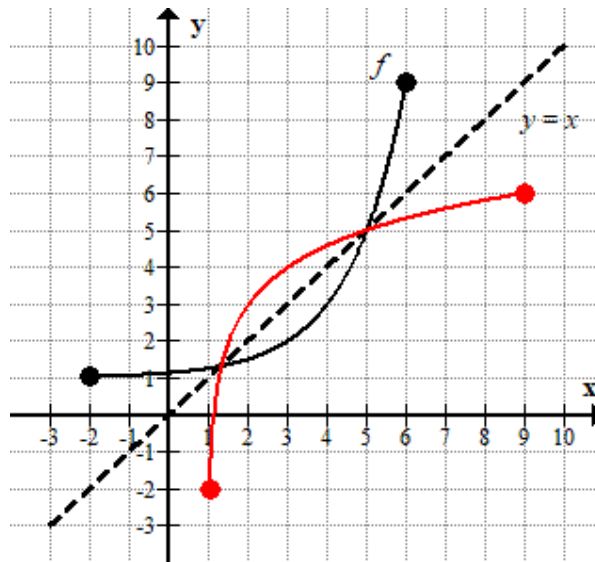
$$32. \quad (a) \quad (f \circ f)(1) = f(f(1)) = f(3) = 2$$

$$(b) \quad (g^{-1} \circ f)(4) = g^{-1}(f(4)) = g^{-1}(1) = 3$$

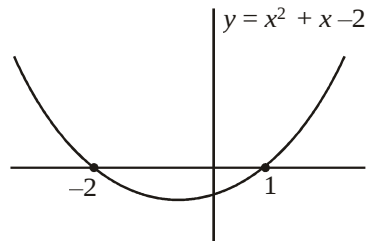
$$(c) \quad f(g(x)) = 1, \text{ so } g(x) = 4, \text{ so } x = 1$$

$$(d) \quad (g^{-1} \circ g)(2) = 2$$

33.



34. (a) $(f \circ g)(x) = \sqrt{x^2 + x - 2}$
 $x^2 + x - 2 \geq 0$



$a = -2, b = 1$

- (b) range is $y \geq 0$

35. (a) in both cases it is the x -coordinate of the vertex.

EITHER by the formula $x = -\frac{b}{2a}$ **OR** by GDC graph, minimum

(i) $a = 2$, (ii) $b = 2$

- (b) **EITHER by solving** $f(x) = 7 \Leftrightarrow 3x^2 - 12x + 7 = 7$ **OR** by GDC graph x -CAL

(i) $f^{-1}(7) = 4$, (ii) $g^{-1}(7) = 0$

36. **METHOD A**

We firstly find $f^{-1}(x) = \sqrt[3]{x+1}$

(a) $g = f^{-1} \circ (f \circ g)$, so $g(x) = \sqrt[3]{2x+2}$

(b) $g = (g \circ f) \circ f^{-1}$, so $g(x) = 2\sqrt[3]{x+1} + 1$

METHOD B

(a) $f(g(x)) = x + 1 \Rightarrow [g(x)]^3 - 1 = 2x + 1$, so $g(x) = \sqrt[3]{2x+2}$

(b) $g(f(x)) = 2x + 1 \Rightarrow g(x^3 - 1) = 2x + 1$

Set $y = x^3 - 1$, then $x = \sqrt[3]{y+1}$, so $g(x) = 2\sqrt[3]{x+1} + 1$

37.

$$g^{-1}(x) = \frac{x+1}{2}$$

$$f(x) = f \circ g \circ g^{-1}(x) = \frac{\frac{x+1}{2} + 1}{2} = \frac{x+3}{4}$$

$$f(x-3) = \frac{(x-3)+3}{4} = \frac{x}{4}$$

38. Let $y = \frac{x^2 - 1}{x^2 + 1} \Rightarrow yx^2 + y = x^2 - 1$

$$x^2(1 - y) = 1 + y \Rightarrow x^2 = \frac{1+y}{1-y} \Rightarrow x = \pm \sqrt{\frac{1+y}{1-y}}$$

Interchanging, $f^{-1}(x) = -\sqrt{\frac{1+x}{1-x}}$

39. (a) By using GDC, graph, x-CAL **OR** SolveN

(i) $f^{-1}(1) = 0$, (ii) $f^{-1}(0) = -0.682$, (iii) $f^{-1}(2) = 0.682$.

(b) Since $f(x) = f^{-1}(x) = x$, we solve instead, $f(x) = x$

EITHER by graph, **OR** by SolveN **OR** analytically $x^3 + x + 1 = x \Leftrightarrow x^3 = -1$

The solution is $x = -1$

B. Exam style questions (LONG)

40. (a) $(f \circ f)(0) = f(1) = -5$

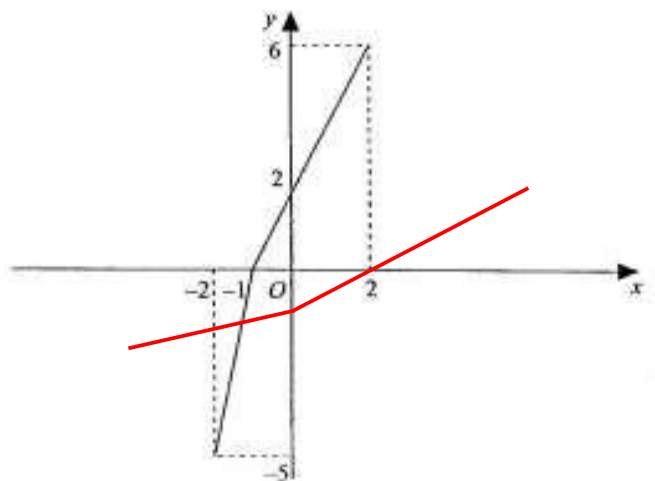
(b) $f(x) = 2(x-2)^2 - 7$

(c) $y \geq -7$

(d) $f^{-1}(x) = 2 - \sqrt{\frac{x+7}{2}}$

(e) Domain: $x \geq -7$, Range: $y \leq 2$

41. (a) We show that $f^{-1}(x) = \frac{3x-1}{x-3}$
- (b) $(f \circ f)(k) = k$
- (c) $(f \circ g)(-2) = f(-5) = \frac{-16}{-8} = 2.$
- (d) $-5 \leq x \leq 6.$



42.

$$\begin{aligned} \text{(a)} \quad f \circ g(x) &= (ax + b + 2)^2 - 3 \\ &= a^2x^2 + abx + 2ax + abx + b^2 + 2b + 2ax + 2b + 4 - 3 \\ &= a^2x^2 + x(2ab + 4a) + (b^2 + 4b + 1) \\ &\Rightarrow a^2x^2 + x(2ab + 4a) + (b^2 + 4b + 1) \equiv 4x^2 + 6x - \frac{3}{4} \end{aligned}$$

Equating coefficients of x^2 gives $a^2 = 4$

$$\Rightarrow a = 2, \text{ since } a > 0$$

Equating coefficients of x gives $2ab + 4a = 6$

$$\Rightarrow 2a(b + 2) = 6$$

$$\Rightarrow 4(b + 2) = 6$$

$$\Rightarrow b = -\frac{1}{2}$$

$$\text{(b)} \quad h \circ k(x) = 5(cx^2 - x + 2) + 2 = 0$$

$$\Rightarrow 5cx^2 - 5x + 10 + 2 = 0$$

$$\Rightarrow 5cx^2 - 5x + 12 = 0$$

Condition for real equal roots is: $b^2 - 4ac = 0$

$$\Rightarrow 25 - 240c = 0$$

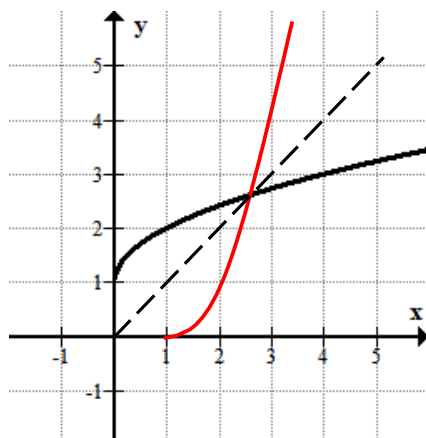
$$\Rightarrow c = \frac{5}{48}$$

43. (a) $x = -1$
- (b) $f^{-1}(x) = (x - 1)^2 - 1 = x^2 - 2x$
- (c) $h(x) = (x^2 - 2x)^2$
- (d) $k(x) = x^4 - 2x^2$

44. (a) $f^{-1}(x) = (x-1)^2$.

(b) $x \geq 1, y \geq 0$

(c)



(d) We solve $f(x) = x$ OR $f^{-1}(x) = x$

$$x = \frac{3 + \sqrt{5}}{2}.$$