## [MAA 1.9] MATHEMATICAL INDUCTION

## SOLUTIONS

### O. Practice questions

**1.** (a) For $n=1$, LHS = $1^3=1$, RHS $=\dfrac{1^2\times 2^2}{4}=1$

For $n=2$, LHS = $1^3+2^3=9$, RHS $=\dfrac{2^2\times 3^2}{4}=9$

For $n=3$, LHS = $1^3+2^3+3^3=36$, RHS $=\dfrac{3^2\times 4^2}{4}=36$

(b) $1^3+2^3+3^3+\cdots+20^3=\dfrac{20^2\times 21^2}{4}=100\times 441=44100$

(c) The inequality takes the form $\dfrac{n^2(n+1)^2}{4}>1000000$

The equation (by using SolveN) gives .

Hence $n=45$

**2.** For $n=1$, it is true since

LHS = $1^3=1$, RHS $=\dfrac{1^2\times 2^2}{4}=1$

Assume it is true for $n=k$,

that is $\displaystyle\sum_{r=1}^{k} r^3=\frac{k^2(k+1)^2}{4}$

Prove it is true for $n=k+1$,

that is $\displaystyle\sum_{r=1}^{k+1} r^3=\frac{(k+1)^2(k+2)^2}{4}$

Indeed,

LHS $\displaystyle=\sum_{r=1}^{k} r^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$ [by assumption]

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2[k^2+4(k+1)]}{4}=\frac{(k+1)^2[k^2+4k+4)]}{4}=\frac{(k+1)^2(k+2)^2}{4}$$ [as required].

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Thus, by mathematical induction, the statement is true for any $n\in\mathbb{Z}^{+}$.

## A-B. Exam style questions (SHORT OR LONG)

### DIVISIBILITY

**3.** For $n=1$, $10^1-1=9$, which is a multiple of 9.

Assume it is true for $n=k$, that is $10^k-1=9a$, with $a\in Z$ (i.e. it is a multiple of 9)

Prove it is true for $n=k+1$, that is $10^{k+1}-1$ is a multiple of 9.

Indeed,

$10^{k+1}-1=10\times 10^k-1=10\times(9a+1)-1$ [by assumption]

$=90a+9=9(10a+1)$ which is a multiple of 9. [as required]

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Thus, by mathematical induction, the statement is true for any $n\in\mathbb{Z}^+$.

**4.** For $n=1$, $1^3+2\times 1=3$, which is a multiple of 3.

Assume it is true for $n=k$, that is $k^3+2k=3a$, with $a\in Z$ (i.e. it is a multiple of 3)

Prove it is true for $n=k+1$, that is $(k+1)^3+2(k+1)$ is a multiple of 3.

Indeed,

$(k+1)^3+2(k+1)=k^3+3k^2+3k+1+2k+2 \quad =k^3+3k^2+5k+3$

$=3a-2k+3k^2+5k+3$ [by assumption]

$=3a+3k^2+3k+3$

$=3(a+k^2+k+1)$ which is a multiple of 3. [as required]

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Thus, by mathematical induction, the statement is true for any $n\in\mathbb{Z}^+$.

**5.** Statement: $5^n+3$ is divisible by 4.

For $n$ = 1, the statement is true, since $5^3+3=8$ is divisible by 4.

We assume that the statement is true for $n = k$, *i.e.* $5^k+3$ is divisible by 4,

say $5^k+3=4a$ (where $a$ is an integer)

We shall show that the statement is true for $n = k+1$, *i.e.* $5^{k+1}+3$ is divisible by 4.

Indeed,

$5^{k+1}+3=5(5^k)+3=5(4a-3)+3$ [by assumption]

$=20a-12=4(5a-3)$ which is divisible by 4. [as required]

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Therefore, by mathematical induction, $5^n+3$ is divisible by 4 for n $\in\mathbb{Z}^+$.

**6.** Statement: $2^{2n} - 3n - 1$ is divisible by 9.

For $n = 1$, the statement is true, since $2^2 - 3 - 1 = 0$ which is divisible by 9.

Assume it is true for $n = k$, *i.e.* $2^{2k} - 3k - 1 = 9a$ (divisible by 9).

We shall show that it is true for $n = k + 1$, i.e. $2^{2(k+1)} - 3(k + 1) - 1$ is divisible by 9.

Indeed,

$$\begin{aligned} 2^{2(k+1)} - 3(k + 1) - 1 &= 4(2^{2k}) - 3k - 4 \\ &= 4(9a + 3k + 1) - 3k - 4 \quad \text{[by assumption]} \\ &= 36a + 12k + 4 - 3k - 4 \\ &= 36a + 9k \\ &= 9(4a + k) \text{ which is divisible by 9.} \end{aligned}$$

The statement is true for $n = 1$ and assuming it is true for $n = k$ it is also true for $n = k + 1$.

Thus, by mathematical induction, the statement is true for any $n \in \mathbb{Z}^+$.

**7.** Let $f(n) = 5^n + 9^n + 2$ and let $P_n$ be the proposition that $f(n)$ is divisible by 4.

Then $f(1) = 16$. So $P_1$ is true

Let $P_n$ be true for $n = k$ ie $f(k)$ is divisible by 4

Consider $f(k + 1) = 5^{k+1} + 9^{k+1} + 2 = 5^k(4 + 1) + 9^k(8 + 1) + 2$

$$= f(k) + 4(5^k + 2 \times 9^k)$$

Both terms are divisible by 4 so $f(k + 1)$ is divisible by 4.

Since $P_1$ is true, and $P_k$ true $\Rightarrow P_{k+1}$ true, $P_n$ is proved true by mathematical induction for $n \in \mathbb{Z}^+$.

**8.** Let $\mathrm{P}(n)$ be the proposition $12^n + 2(5^{n-1})$ is a multiple of 7

When $n = 1$ $\quad 12^1 + 2(5^0) = 14$ (which is a multiple of 7)

$\therefore \mathrm{P}(1)$ is true

Assume $\mathrm{P}(k)$ is true,

*i.e.* $12^k + 2(5^{k-1}) = 7m,\ m \in \mathbb{Z}^+$

For $n = k + 1$

$$\begin{aligned} 12^{k+1} + 2(5^k) &= 12(12^k) + 2(5^k) \\ &= 12(7m - 2 \times 5^{k-1}) + 2 \times 5^k \\ &= 12(7m - 2 \times 5^{k-1}) + 10 \times 5^{k-1} \\ &= 12(7m) - 24(5^{k-1}) + 10(5^{k-1}) \\ &= 7(12m - 2 \times 5^{k-1}) \end{aligned}$$

$\therefore 12^{k+1} + 2(5^k)$ is a multiple of 7

So $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true, and $\mathrm{P}(1)$ is true.

Hence by induction $\mathrm{P}(n)$ is true.

## SERIES

**9.** For $n=1$, the statement is true since

LHS = $\sum_{r=1}^{1} r^2 = 1^2 = 1$, RHS = $\frac{1\times 2\times 3}{6} = 1$

Assume it is true for $n=k$, that is $\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$

Prove it is true for $n=k+1$, that is $\sum_{r=1}^{k+1} r^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Indeed,

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{[by assumption]}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$= \frac{(k+1)\left[2k^2+k+6k+6\right]}{6}$$

$$= \frac{(k+1)\left[2k^2+7k+6\right]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

[**hint**: even if you cannot see it, write it down, since this is what you expect!]

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Therefore, by mathematical induction, $5^n + 3$ is divisible by 4 for n $\in \mathbb{Z}^+$.

**10.** Similar

**11.** For $n$ =1,

LHS $= a$ **and** RHS $= \frac{a(1-r)}{1-r} = a$

Hence the result is true for $n=1$

Assume it is true for $n=k$

$$a + ar + ar^2 + \ldots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$$

Consider $n=k+1$:

$$a + ar + ar^2 + \ldots + ar^{k-1} + ar^k = \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r}$$

The result is true for $n=k \Rightarrow$ it is true for $n=k+1$ **and** as it is true for $n=1$, the result is proved by mathematical induction.

**12.** Let P($n$) be the proposition for any $n$.

For $n=1$

$(1+1)2^{1-1}=2(1)=2$

$(1)\times 2^1=2$

Therefore $\mathrm{P}(1)$ is true.

Assume $\mathrm{P}(k)$ is true i.e. $\sum_1^k (r+1)2^{r-1}=k\times 2^k$

Then $\sum_1^{k+1}(r+1)2^{r-1}=\sum_1^k(r+1)2^{r-1}+(k+1+1)2^{k+1-1}$

$=k\times 2^k+(k+2)2^k$

$=2^{k+1}(k+1)$

Therefore,

$\mathrm{P}(k)$ true $\Rightarrow \mathrm{P}(k+1)$ true, since $\mathrm{P}(1)$ is true, then $\mathrm{P}(n)$ is true.

**13.** **(a)** Let P($n$) be the proposition for any $n$.

$\mathrm{P}(1): \sum_1^1 \frac{1}{(2r-1)(2r+1)}=\frac{1}{3}=\frac{1}{2(1)+1}$ so $\mathrm{P}(1)$ is true

Assume that $\mathrm{P}(k)$ is true

$$\mathrm{P}(k+1): \sum_1^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{k}{(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}$$

$$=\frac{(k+1)}{(2k+1)+1}$$

Therefore $\mathrm{P}(1)$ is true and $\mathrm{P}(k)\Rightarrow \mathrm{P}(k+1)$ so $\mathrm{P}(n)$ is true $\forall n\in\mathbb{Z}^+$.

(b) Checking that $\frac{1}{3}+\frac{1}{15}+\frac{1}{35}$ is the same as $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$

(e.g. substitute $r=1,2$)

Sum is therefore sum of $(n+1)$ terms

i.e. $\frac{(n+1)}{2(n+1)+1}$

$=\frac{n+1}{2n+3}$

**14.**

(a) If $n=1$, then $(1)(1!)=(1+1)!-1$ is true

Assume true for $n=k$

$\Rightarrow (1)(1!)+(2)(2!)+...+(k)(k!)=(k+1)!-1$

Add the next term $(k+1)(k+1)!$ to both sides

$(1)(1!)+(2)(2!)+...+(k)(k!)+(k+1)(k+1)!=(k+1)!-1+(k+1)(k+1)!$

$=(k+1)![1+k+1]-1$

$=(k+2)!-1$

True for $k\Rightarrow$ True for $k+1$ and since true for $n=1$, result proved by mathematical induction.

(b) $(n+1)!-1>1000000000$

$(n+1)!>1000000001$

from GDC minimum value of $n=12$

**15.** Statement: $u_n = u_1 + (n-1)d$

For $n = 1$, the statement is true since LHS = $u_1$, RHS = $u_1 + 0d = u_1$.

Assume it is true for $n = k$, that is $u_k = u_1 + (k-1)d$

Prove it is true for $n = k+1$, that is $u_{k+1} = u_1 + kd$

Indeed,

$u_{k+1} = u_k + d = u_1 + (k-1)d + d = u_1 + kd - d + d = u_1 + kd$

The statement is true for $n = 1$ and assuming it is true for $n = k$ it is also true for $n = k+1$.

Therefore, by mathematical induction the statement is true for any integer $n \in Z^+$.

**16.** Similar.

**17.** Given: $u_1 = 10$, $u_{n+1} = 2u_n + 2$

Statement: $u_n = 3(2)^{n+1} - 2$

For $n = 1$, the statement is true since

LHS = $u_1 = 10$, RHS = $3(2)^2 - 2 = 10$.

Assume it is true for $n = k$, that is $u_k = 3(2)^{k+1} - 2$

Prove it is true for $n = k+1$, that is $u_{k+1} = 3(2)^{k+2} - 2$

Indeed,

$u_{k+1} = 2u_k + 2 = 2(3(2)^{k+1} - 2) + 2$ [by assumption]

$= 3(2)^{k+2} - 4 + 2$

$= 3(2)^{k+2} - 2$

The statement is true for $n = 1$ and assuming it is true for $n = k$ it is also true for $n = k+1$.

Therefore, by mathematical induction the statement is true for any integer $n \in Z^+$.

**18.** Given: $u_1 = 5$, $u_2 = 8$, $u_{n+1} = 2u_n - u_{n-1}$

Statement: $u_n = 3n + 2$ **[notice: we need two first steps and two assumptions]**

For $n = 1$, the statement is true since LHS = $u_1 = 5$, RHS = $3 + 2 = 5$.

For $n = 2$, the statement is true since LHS = $u_1 = 8$, RHS = $6 + 2 = 8$.

Assume it is true for $n = k-1$ and $n = k$, that is $u_{k-1} = 3(k-1) + 2$ and $u_k = 3k + 2$

Prove it is true for $n = k+1$, that is $u_{k+1} = 3(k+1) + 2 = 3k + 5$

Indeed,

$u_{n+1} = 2u_k - u_{k-1} = 2\left[3k + 2\right] - \left[3(k-1) + 2\right]$ [by assumption]

$= 6k + 4 - 3k + 3 - 2$

$= 3k + 5$ [as required]

The statement is true for $n = 1$ and $n = 2$ and assuming it is true for $n = k-1$ and $n = k$, it is also true for $n = k+1$.

Therefore, by mathematical induction the statement is true for any integer $n \in Z^+$.

## INEQUALITIES

**19.** - For $n=3$ the statement is true since

LHS = $4!=24$ RHS = $2^3 3=24$

- Assume it is true for $n=k$ ($k \geq 3$), i.e. $(k+1)! \geq 2^k k$
- Prove it is true for $n=k+1$, i.e. $(k+2)! \geq 2^{k+1}(k+1)$

Indeed,

$$(k+2)! = (k+1)!(k+2) \geq 2^k k(k+2) \qquad \text{[by assumption]}$$

It suffices to show that

$$2^k k(k+2) \geq 2^{k+1}(k+1)$$

$$\Leftrightarrow \quad k(k+2) \geq 2(k+1)$$

$$\Leftrightarrow \quad k^2+2k \geq 2k+2$$

$$\Leftrightarrow \quad k^2 \geq 2$$

which is true since $k \geq 3$

The statement is true for $n=3$ and assuming it is true for $n=k$ it is also true for $n=k+1$. Therefore, by mathematical induction the statement is true for any integer $n \geq 3$.

**20.** - For $n=2$ the statement is true since

LHS = $3^2=9$ RHS = $2^2+4=8$ and 9>8

- Assume it is true for $n=k$ ($k \geq 2$), i.e. $3^k > k^2+2k$
- Prove it is true for $n=k+1$, i.e. $3^{k+1} > (k+1)^2+2(k+1)$

Indeed,

$$3^{k+1} = 3 \cdot 3^k > 3(k^2+2k) \qquad \text{[by assumption]}$$

It suffices to show that

$$\Leftrightarrow \quad 3(k^2+2k) \geq (k+1)^2+2(k+1)$$

$$\Leftrightarrow \quad 3k^2+6k \geq k^2+2k+1+2k+2$$

$$\Leftrightarrow \quad 2k^2+2k \geq 3$$

which is true since $k \geq 2$

The statement is true for $n=2$ and assuming it is true for $n=k$ it is also true for $n=k+1$. Therefore, by mathematical induction the statement is true for any integer $n \geq 2$.

## TRIGONOMETRY

**21.** The statement is $(\cos x)(\cos 2x)\cdots(\cos 2^{n-1}x) = \dfrac{\sin 2^n x}{2^n \sin x}$

For $n=1$, the statement is true since

LHS = $\cos x$, RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x \cos x}{2\sin x} = \cos x$

Assume it is true for $n=k$, that is

$$(\cos x)(\cos 2x)\cdots(\cos 2^{k-1}x) = \frac{\sin 2^k x}{2^k \sin x}$$

Prove it is true for $n=k+1$, that is

$$(\cos x)(\cos 2x)\cdots(\cos 2^{k-1}x)(\cos 2^k x) = \frac{\sin 2^{k+1}x}{2^{k+1}\sin x}$$

Indeed,

LHS $= \dfrac{\sin 2^k x}{2^k \sin x}(\cos 2^k x) = \dfrac{2\times\sin 2^k x\cos 2^k x}{2\times 2^k \sin x} = \dfrac{\sin 2\times 2^k x}{2^{k+1}\sin x} = \dfrac{\sin 2^{k+1}x}{2^{k+1}\sin x}$

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Therefore, by mathematical induction the statement is true for any integer $n \in Z^+$.

**22.** (a) RHS to LHS:

$\sin((2n+1)x)\ \cos x - \cos((2n+1)x)\ \sin x = \sin((2n+1)x - x)$

$= \sin(2nx + x - x) = \sin 2nx$

(b) For $n=1$, the statement is true since

LHS = $\cos x$, RHS $= \dfrac{\sin 2x}{2\sin x} = \dfrac{2\sin x\cos x}{2\sin x} = \cos x$

Assume it is true for $n=k$, that is

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos((2k-1)x) = \frac{\sin 2kx}{2\sin x}$$

Prove it is true for $n=k+1$, that is

$$\cos x + \cos 3x + \cos 5x + \cdots + \cos((2k-1)x) + \cos((2k+1)x) = \frac{\sin((2k+2)x)}{2\sin x}$$

Indeed,

LHS $= \dfrac{\sin 2kx}{2\sin x} + \cos((2k+1)x) = \dfrac{\sin 2kx + 2\cos((2k+1)x)\sin x}{2\sin x}$

$$= \frac{\sin((2k+1)x)\ \cos x - \cos((2k+1)x)\ \sin x + 2\cos((2k+1)x)\sin x}{2\sin x}$$

$$= \frac{\sin((2k+1)x)\ \cos x + \cos((2k+1)x)\sin x}{2\sin x}$$

$$= \frac{\sin((2k+1)x + x)}{2\sin x}$$

$$= \frac{\sin((2k+2)x)}{2\sin x}$$

The statement is true for $n = 1$ and assuming it is true for $n = k$ it is also true for $n = k+1$.

Therefore, by mathematical induction the statement is true for any integer $n \in Z^+$.

**23.** (a) RHS to LHS:

$$\cos((2n+1)x)\ \cos x + \sin((2n+1)x)\ \sin x = \cos((2n+1)x - x)$$

$$= \cos(2nx + x - x) = \cos 2nx$$

(b) For $n = 1$, the statement is true since

LHS = $\sin x$, RHS $= \dfrac{1-\cos 2x}{2\sin x} = \dfrac{1-1+2\sin^2 x}{2\sin x} = \dfrac{2\sin^2 x}{2\sin x} = \sin x$

Assume it is true for $n = k$, that is

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin((2k-1)x) = \frac{1-\cos 2kx}{2\sin x}$$

Prove it is true for $n = k+1$, that is

$$\sin x + \sin 3x + \sin 5x + \cdots + \sin((2k-1)x) + \sin((2k+1)x) = \frac{1-\cos((2k+2)x)}{2\sin x}$$

Indeed,

LHS $= \dfrac{1-\cos 2kx}{2\sin x} + \sin((2k+1)x) = \dfrac{1-\cos 2kx + 2\sin((2k+1)x)\sin x}{2\sin x}$

$$= \frac{1-\cos((2k+1)x)\ \cos x - \sin((2k+1)x)\ \sin x + 2\sin((2k+1)x)\sin x}{2\sin x}$$

$$= \frac{1-\cos((2k+1)x)\ \cos x + \sin((2k+1)x)\sin x}{2\sin x}$$

$$= \frac{1-[\cos((2k+1)x)\ \cos x - \sin((2k+1)x)\sin x]}{2\sin x}$$

$$= \frac{1-\cos((2k+1)x + x)}{2\sin x}$$

$$= \frac{1-\cos((2k+2)x)}{2\sin x}$$

The statement is true for $n = 1$ and assuming it is true for $n = k$ it is also true for $n = k+1$.

Therefore, by mathematical induction the statement is true for any integer $n \in Z^+$.

## DERIVATIVES

**24.** For $n=1$, the statement is true since

LHS = $f'(x)=\dfrac{1}{x}$, RHS $=(-1)^0\dfrac{0!}{x^1}=\dfrac{1}{x}$

Assume it is true for $n=k$, that is $f^{(k)}(x)=(-1)^{k-1}\dfrac{(k-1)!}{x^k}$

Prove it is true for $n=k+1$, that is $f^{(k+1)}(x)=(-1)^k\dfrac{k!}{x^{k+1}}$

Indeed,

$$f^{(k+1)}(x)=\left[f^{(k)}(x)\right]'=\left[(-1)^{k-1}(k-1)!\,x^{-k}\right]' \quad \text{[by assumption]}$$

$$=(-1)^{k-1}(k-1)!(-k)\,x^{-k-1}$$

$$=(-1)^k\frac{k!}{x^{k+1}} \quad \text{[as required]}$$

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Therefore, by mathematical induction the statement is true for any integer $n\in Z^+$.

**25.** (a) $f'(x)=3e^{3x}$, $f'(x)=9e^{3x}$, $f'(x)=27e^{3x}$

Guess: $f^{(n)}(x)=3^n e^{3x}$

(b) Similar to the previous exercise.

**26.** (a) $f'(x)=\dfrac{1}{x+1}$, $f'(x)=-\dfrac{1}{(x+1)^2}$, $f'(x)=\dfrac{2}{(x+1)^3}$, $f^{(4)}(x)=-\dfrac{6}{(x+1)^4}$

(b) $f^{(n)}(x)=(-1)^{n-1}\dfrac{(n-1)!}{(x+1)^n}$

(c) For $n=1$, LHS = $f'(x)=\dfrac{1}{x+1}$, RHS $=(-1)^0\dfrac{0!}{(x+1)^1}=\dfrac{1}{x+1}$

Assume it is true for $n=k$, that is $f^{(k)}(x)=(-1)^{k-1}\dfrac{(k-1)!}{(x+1)^k}$

Prove it is true for $n=k+1$, that is $f^{(k+1)}(x)=(-1)^k\dfrac{k!}{(x+1)^{k+1}}$

Indeed,

$$\text{LHS} = f^{(k+1)}(x)=\left[f^{(k)}(x)\right]'=-(-1)^{k-1}(k-1)!\,\frac{k!}{(x+1)^{k+1}} \quad \text{[by assumption]}$$

$$=(-1)^k\frac{k!}{(x+1)^{k+1}} \quad \text{[as required].}$$

The statement is true for $n=1$ and assuming it is true for $n=k$ it is also true for $n=k+1$.

Thus, by mathematical induction, the statement is true for any $n\in \blacksquare^{+}$.

**27.** Let $p_n$ be the statement $\frac{\mathrm{d}}{\mathrm{d}x} = nx^{n-1}$ for all positive integer values of $n$.

If $n = 1$ then $\frac{\mathrm{d}}{\mathrm{d}x}(x^1) = \lim_{k\to 0}\left(\frac{(x+k)-x}{k}\right) = 1 = 1x^0$

Assume the formula is true for $n = k$, that is, $\frac{\mathrm{d}}{\mathrm{d}x}(x^k) = kx^{k-1}$.

Then $\frac{\mathrm{d}}{\mathrm{d}x}(x^{k+1}) = \frac{\mathrm{d}}{\mathrm{d}x}(x \times x^k)$

$= kx^{k-1} \times x + x^k$ (using the results for $n = k$ and $n = 1$ given above)

$= x^k(k + 1)$

which is the formula when $n = k + 1$

So if the formula is true for $n = k$ then it is true for $n = k + 1$.

$p_1$ is true, therefore $p_n$ is true for all integer values of $n$.

**28.** Let $p_n$ be the statement $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\cos x = \cos\left(x + \frac{n\pi}{2}\right)$ for all positive integer values of $n$.

For $n = 1$,

LHS $= \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$

RHS $= \cos\left(x + \frac{\pi}{2}\right) = -\sin x$

Therefore $p_1$ is true.

Assume the formula is true for $n = k$,

that is, $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\cos x) = \cos\left(x + \frac{k\pi}{2}\right)$

Prove the formula is true for $n = k+1$,

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x}(\cos x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(x + \frac{k\pi}{2}\right)\right)$$

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\cos x) = -\sin\left(x + \frac{kx}{2}\right)$$

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\cos x) = \cos\left(x + \frac{kx}{2} + \frac{\pi}{2}\right)$$

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(\cos x) = \cos\left(x + \frac{(k+1)\pi}{2}\right)$$

which is $p_n$ when $n = k + 1$.

(So if $p_n$ is true for $n = k$ then it is true for $n = k + 1$ and by the principle of mathematical induction $p_n$ is true for all positive integer values of $n$.)

**29.** (a) (i) $f'(x) = p\mathrm{e}^{px}(x+1) + \mathrm{e}^{px} = \mathrm{e}^{px}\left(p(x+1)+1\right)$

(ii) The result is true for $n=1$ since

$\text{LHS} = \mathrm{e}^{px}\left(p(x+1)+1\right)$

$\text{RHS} = p^{1-1}\mathrm{e}^{px}\left(p(x+1)+1\right) = \mathrm{e}^{px}\left(p(x+1)+1\right)$.

Assume true for $n=k$ : $f^{(k)}(x) = p^{k-1}\mathrm{e}^{px}\left(p(x+1)+k\right)$

Prove true for n = $k+1$: $f^{(k+1)}(x) = p^{k}\mathrm{e}^{px}\left(p(x+1)+k+1\right)$

Indeed,

$$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = p^{k-1}p\mathrm{e}^{px}\left(p(x+1)+k\right) + p^{k-1}\mathrm{e}^{px}p$$

$$= p^{k}\mathrm{e}^{px}\left(p(x+1)+k+1\right)$$

Therefore, true for $n=k \Rightarrow$ true for $n=k+1$ and the proposition is proved by induction.

**30.**

(a) $y = x\mathrm{e}^{-x}$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x} - x\mathrm{e}^{-x} \quad = \mathrm{e}^{-x}(1-x)$$

(b) When $n=1$, the formula gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (-1)^2\mathrm{e}^{-x}(1-x)$

$$= \mathrm{e}^{-x}(1-x)$$

as demonstrated in part (a)
Assume true for $n=k$

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^{k+1}\mathrm{e}^{-x}(k-x)$$

Consider, $\dfrac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^{k+1}\mathrm{e}^{-x}(k-x)\right]$

$$= (-1)^{k+1}\left[-\mathrm{e}^{-x}(k-x) - \mathrm{e}^{-x}\right]$$

$$= (-1)^{k+2}\left[\mathrm{e}^{-x}(k+1-x)\right]$$

Hence if it is true for $k$, then it is true for $k+1$
Since the result is true for $n=1$ and $\mathrm{P}(k) \Rightarrow \mathrm{P}(k+1)$ the result is proved by mathematical induction $\forall n \in \mathbb{Z}^{+}$.

**31.**

Show true for $n = 1$

$f'(x) = e^{2x} + 2xe^{2x}$

$\quad = e^{2x}(1+2x) = (2x + 2^0)e^{2x}$

Assume true for $n = k$, *i.e.* $f^{(k)}x = (2^k x + k \times 2^{k-1})e^{2x}$, $k \geq 1$

Consider $n = k+1$, *i.e.* an attempt to find $\frac{d}{dx}\left(f^k(x)\right)$.

$f^{(k+1)}(x) = 2^k e^{2x} + 2e^{2x}(2^k x + k \times 2^{k-1})$

$\quad = \left(2^k + 2(2^k x + k \times 2^{k-1})\right)e^{2x}$

$\quad = (2 \times 2^k x + 2^k + k \times 2 \times 2^{k-1})e^{2x}$

$\quad = (2^{k+1} x + 2^k + k \times 2^k)e^{2x}$

$\quad = \left(2^{k+1} x + (k+1)2^k\right)e^{2x}$

$P(n)$ is true for $k \Rightarrow P(n)$ is true for $k+1$, and since true for $n = 1$, result proved by mathematical induction $\forall n \in \mathbb{Z}^+$

**32.** For $n = 1$ the statement is true since

LHS = $f'(x) = 2\cos 2x \Rightarrow f''(x) = -4\sin 2x$

RHS = $= (-4)^1 \sin 2x = -4\sin 2x$

Assume it is true for $n = k$ i.e. $f^{(2k)}(x) = (-4)^k \sin 2x$

Prove it is true for $n = k+1$, i.e. $f^{(2k+2)}(x) = (-4)^{k+1} \sin 2x$

Indeed,

$$f^{(2k+2)}(x) = \left\lfloor \left\lceil f^{(2k)}(x) \right\rceil \right\rfloor'' = \left\lceil \left\lfloor (-4)^k 2\cos 2x \right\rfloor \right\rceil' = -(-4)^k 4\sin 2x \;= (-4)^{k+1} \sin 2x$$

The statement is true for $n = 1$ and assuming it is true for $n = k$ it is also true for $n = k+1$. Therefore, by mathematical induction the statement is true for any integer $n$.

## COMPLEX NUMBERS

**33.** (a) $(1 + i)^2 = 1 + 2i + i^2 = 2i$

(b) $(1 + i)^{4n}$

Let P($n$) be the proposition: $(1 + i)^{4n} = (-4)^n$

We must first show that P(1) is true.

$(1 + i)^4 = ((1 + i)^2)^2 = (2i)^2 = 4(i)^2 = (-4)^1$

Next, assume that for some $k$ e $\mathbb{N}^+$

P($k$) is true, then show that P($k$ +1) is true.

P($k$):$(1 + i)^{4k} = (-4)^k$

Now, $(1 + i)^{4(k+1)} = (1 + i)^{4k}(1 + i)^4$

$= (-4)^k(-4)$

$= (-4)^{k+1}$

Therefore, by mathematical induction P($n$) is true for all $n \in \mathbb{N}^+$

(c) $(1 + i)^{32} = (1 + i)^{4(8)} = (-4)^8 = 65536$

**34.** The result is true for $n = 1$, since

$$\text{LHS} = \cos\theta + \text{i}\sin\theta$$

and $$\text{RHS} = \cos\theta + \text{i}\sin\theta$$

Let the proposition be true for $n = k$.

$$(\cos\theta + \text{i}\sin\theta)^k = \cos(k\theta) + \text{i}\sin(k\theta)$$

Now show $n = k$ true implies $n = k + 1$ also true.

$$(\cos\theta + \text{i}\sin\theta)^{k+1} = (\cos\theta + \text{i}\sin\theta)^k(\cos\theta + \text{i}\sin\theta)$$

$$= (\cos(k\theta) + \text{i}\sin(k\theta)(\cos\theta + \text{i}\sin\theta)$$

$$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \text{i}(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta)$$

$$= \cos(k\theta + \theta) + \text{i}\sin(k\theta + \theta)$$

$$= \cos(k + 1)\theta + 1\sin(k + 1)\theta \Rightarrow n = k + 1 \text{ is true.}$$

Therefore, by mathematical induction statement is true for $n \geq 1$.

**35.**

(Let $\text{P}(n)$ be $(1+\sqrt{3}\text{i})^n = 2^n\left(\cos\frac{n\pi}{3} + \text{i}\sin\frac{n\pi}{3}\right)$)

For $n = 1$: $2^1\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right) = 1 + \sqrt{3}\text{i}$, so $\text{P}(1)$ is true

Assume $\text{P}(k)$ is true,

$$(1+\sqrt{3}\text{i})^k = 2^k\left(\cos\frac{k\pi}{3} + \text{i}\sin\frac{k\pi}{3}\right)$$

Consider $\text{P}(k+1)$

$$(1+\sqrt{3}\text{i})^{k+1} = (1+\sqrt{3}\text{i})^k(1+\sqrt{3}\text{i})$$

$$= 2^k\left(\cos\frac{k\pi}{3} + \text{i}\sin\frac{k\pi}{3}\right)2\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)$$

$$= 2^{k+1}\left(\cos\frac{(k+1)\pi}{3} + \text{i}\sin\frac{(k+1)\pi}{3}\right)$$

$\text{P}(k)$ true implies $\text{P}(k+1)$ true, $\text{P}(1)$ true so $\text{P}(n)$ true $\forall n \in \mathbb{Z}^+$.