INTERNATIONAL BACCALAUREATE

Mathematics: analysis and approaches

MAA

EXERCISES [MAA 1.8]

METHODS OF PROOF

O. Practice questions

1. [Maximum mark: 6] ***[without GDC]***

It is known that for an integer a, if a^2 is a multiple of 3 then a is also a multiple of 3.

Use this fact and contradiction to show that $\sqrt{3}$ is irrational.

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2. [Maximum mark: 8] ***[without GDC]***

Prove the following statements by using contradiction

(a) There exist no integers a and b for which $6a+2b=35$ [2]

(b) There exist no integers a and b for which $6a+9b=35$ [2]

(c) The equation $2x^5-6x^2+14x=1$ has no integer solution. [2]

(d) The equation $3x^5+15x^2-14=0$ has no integer solution. [2]

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3. [Maximum mark: 12] ***[without GDC]***

Rational numbers, by definition, have the form of a fraction $\frac{a}{b}$ where $a \in Z$ and $b \in Z^*$.

(a) Given that x and y are rational numbers, prove that

(i) $x+y$ is a rational number. (ii) xy is a rational number. [4]

(b) Given that x and y are irrational numbers, their sum $x+y$ and their product xy are usually irrational as well. Peter believes that

(i) $x+y$ is always irrational. (ii) xy is always irrational.

Use **counterexamples** to prove that both Peter's claims are false. [4]

(c) Given that x is rational and y irrational, use **contradiction** to prove that

(i) $x+y$ is irrational (ii) xy is irrational, provided that $x \neq 0$ [4]

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A. Exam style questions (SHORT)

4. [Maximum mark: 7] ***[without GDC]***

It is known that

the product of even integers is even

the product of odd integers is odd.

Prove by using contradiction that

(a) for any integer a, if a^3 is even then a is even. [2]

(b) $\sqrt[3]{2}$ is irrational. [5]

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5. [Maximum mark: 7] ***[without GDC]***

Given that π is irrational, show that $2\pi + 1$ is also irrational.

6. [Maximum mark: 7] ***[without GDC]***

The number $\log_a b$ is defined as follows

$$\log_a b = x \text{ if and only if } a^x = b$$

Show that the number $\log_2 3$ is irrational.

7. [Maximum mark: 8] ***[without GDC]***

(a) Prove the identity $2x^3 + 7x^2 - 14x + 5 \equiv (x+5)(2x^2 - 3x + 1)$ [2]

(b) Given that $3x^3 + 13x^2 - 3x + 35 \equiv (x+5)(ax^2 + bx + c)$ find the values of a, b, c. [3]

(c) Given that $ax^3 + bx^2 - 23x + c \equiv (x+5)(2x^2 + dx + 2)$ find the values of a, b, c, d. [3]

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8. [Maximum mark: 6] ***[without GDC]***

Prove the following statements:

(a) If n is an odd integer then n^2+2n+5 is even. [4]

(b) For all integers n, if n^2+2n+5 is odd then n is even. [2]

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9. [Maximum mark: 4] ***[without GDC]***

(a) For any $x, y, z \in R$, prove that $x^2+y^2+z^2=0 \Rightarrow xyz=0$ [2]

(b) Is the converse of (a) true? Justify your answer. [2]

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10. [Maximum mark: 6] ***[without GDC]***

Suppose $a, b, c \in Z$. Prove by contradiction that

(a) If $a + b \geq 19$, then $a \geq 10$ or $b \geq 10$ [3]

(b) If $a + b + c \geq 19$, then $a \geq 7$ or $b \geq 7$ or $c \geq 7$ [3]

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11. [Maximum mark: 4] ***[without GDC]***

Suppose a, b are positive real numbers. Prove the statement

If $ab = c$ then $a \leq \sqrt{c}$ or $b \leq \sqrt{c}$.

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12. [Maximum mark: 4] ***[without GDC]***

Suppose $a, b \in Z$. Given that $5a + 3b \geq 81$, prove that $a \geq 11$ or $b \geq 11$.

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13*. [Maximum mark: 6] ***[without GDC]***

(a) Suppose that $\sum_{i=1}^{9} x_i \geq 46$, where all x_i are integers. Prove by contradiction that at least one of the values x_i is greater than or equal to 6. [3]

(b) Suppose that $\sum_{i=1}^{9} x_i > 45$, where all x_i are real numbers. Prove by contradiction that at least one of the values x_i is greater than 5. [3]

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B. Exam style questions (LONG)

14. [Maximum mark: 10] ***[without GDC]***

Integers divided by 3 leave a remainder 0, 1 or 2; thus any integer has one of the forms

$3n$	(multiples of 3)
$3n+1$	(leave remainder 1)
$3n+2$	(leave remainder 2)

(a) Prove that the sum of two multiples of 3 is a multiple of 3. [2]

(b) For two numbers that are not multiples of 3

Ann claims that their sum is always a multiple of 3

Bill claims that their sum is always a non-multiple of 3

Show that both claims are wrong. [3]

(c) Prove that for any integer a, the number a^2 cannot be of the form $3n+2$. [5]

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15. [Maximum mark: 12] ***[without GDC]***

Consider the equation of integers $a+b+c=d$.

(a) Use a deductive proof to prove the statements

(i) "if a, b, c are all even then d is also even";

(ii) "if a, b, c are all odd then d is also odd" [5]

(b) Use a counterexample to disprove the statement

"if d is odd then a, b, c are all odd". [2]

(c) State whether the following statement is true or false and prove your claim.

"if d is even then a, b, c are all even". [3]

(d) Use contradiction to prove the statement

"if d is even then at least one of a, b, c is even". [2]

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16*. [Maximum mark: 12] ***[without GDC]***

(a) Prove the identity $a^5 - 1 \equiv (a-1)(a^4 + a^3 + a^2 + a + 1)$ [2]

(b) By letting $a = \dfrac{x}{y}$ deduce an identity for $x^5 - y^5$ with no fractions. [3]

(c) Write down similar identities for $a^n - 1$ and $x^n - y^n$, $n \in Z^+$ [4]

(d) Replace y with $-y$ in the last identity to deduce a similar result for $x^n + y^n$, where *n* is odd. [3]

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17*. [Maximum mark: 10] ***[without GDC]***

Let $f(x) = 2x + 3$, $g(x) = \dfrac{x+3}{2x-4}$ and $h(x) = x^2 + 1$

(a) Show that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ [2]

(b) Show that $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$ [4]

(c) Show that $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$ [2]

(d) Show that $x_1 \neq x_2 \not\Rightarrow h(x_1) \neq h(x_2)$ by using a counterexample. [2]

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18*. [Maximum mark: 10] ***[without GDC]***

In a birthday party there are **50 people** who may give handshakes to each other

(a) Show that there is always a couple of people who give the same number of handshakes. [5]

Each person is asked to write on a piece of paper their favorite day of the weak.

(b) Show that at least one of the days will appear at least 8 times. [5]