

[MAA 1.6] BINOMIAL THEOREM

SOLUTIONS

O. Practice questions

1. (a) $(1 \pm x)^3 = 1 \pm 3x + 3x^2 \pm x^3$
 (b) $(1 \pm x)^4 = 1 \pm 4x + 6x^2 \pm 4x^3 + x^4$
 (c) $(1 \pm x)^5 = 1 \pm 5x + 10x^2 \pm 10x^3 + 5x^4 \pm x^5$
2. (a) $(1 + x)^{10} = 1 + 10x + 45x^2 + 120x^3 + \dots$
 (b) $(1 + 2x)^{10} = 1 + 20x + 180x^2 + 960x^3 + \dots$
 (c) $(2 - x)^{10} = 1024 - 5120x + 11520x^2 - 15360x^3 + \dots$
3. $-120x^7$
4. $45x^{16}$
5. (a) (i) $\left(x - \frac{1}{x}\right)^3 = x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$ (ii) $\left(x - \frac{1}{x}\right)^4 = x^4 - 4x^2 + 6 - \frac{4}{x^2} - \frac{1}{x^4}$
 (b) (i) $-\binom{10}{5} = -252$ (ii) $\binom{10}{4} = 210$ (iii) $\binom{10}{6} = 210$
6. (a) $(1 + 2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$ (b) $8x^2$ (c) $32x^2$ (d) $71x^2$
7. (a) (show) (b) (i) $\binom{10}{n}$ (ii) 45 (iii) 120
8. (a) (show) (b) $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{6} = 120$
9. (a) 43110144 (b) 51963120
 (c) $531441x^{12} - 4251528x^{10} + 15588936x^8 - 34642080x^6 + \dots$
10. (a) 10264320 (b) 43110144 (c) $531441x^{24} - 425152x^{21} + 15588936x^{18} + \dots$
11. (a) -3041280 (b) 10264320 (c) $531441x^{36} - 425152x^{32} + 15588936x^{28} + \dots$

A. Exam style questions (SHORT)

12. $(\sqrt{3} + \sqrt{2})^3 = \dots = 9\sqrt{3} + 11\sqrt{2}$
13. $(\sqrt{3} + \sqrt{2})^4 = \dots = 49 + 20\sqrt{6}$
14. $(\sqrt{3} - 2)^3 = (\sqrt{3})^3 + 3(\sqrt{3})^2(-2)^1 + 3(\sqrt{3})(-2)^2 + (-2)^3$
 $= 3\sqrt{3} + (-18) + 12\sqrt{3} - 8$
 $= 15\sqrt{3} - 26$ (accept $a = 15$, $b = -26$)

15. METHOD 1

Using binomial expansion

$$(3 + \sqrt{7})^3 = 3^3 + 3 \times 3^2 \times \sqrt{7} + 3 \times 3 \times (\sqrt{7})^2 + (\sqrt{7})^3 = 27 + 27\sqrt{7} + 63 + 7\sqrt{7}$$

$$(3 + \sqrt{7})^3 = 90 + 34\sqrt{7} \quad (\text{so } p = 90, q = 34)$$

METHOD 2

For multiplying

$$(3 + \sqrt{7})^2 (3 + \sqrt{7}) = (9 + 6\sqrt{7} + 7)(3 + \sqrt{7}) = 27 + 9\sqrt{7} + 18\sqrt{7} + 42 + 21 + 7\sqrt{7}$$

$$(3 + \sqrt{7})^3 = 90 + 34\sqrt{7} \quad (\text{so } p = 90, q = 34)$$

16. (a)
$$\left(e + \frac{1}{e}\right)^4 = \binom{4}{0}e^4 + \binom{4}{1}e^3\left(\frac{1}{e}\right) + \binom{4}{2}e^2\left(\frac{1}{e}\right)^2 + \binom{4}{3}e\left(\frac{1}{e}\right)^3 + \binom{4}{4}\left(\frac{1}{e}\right)^4$$

$$\left(e + \frac{1}{e}\right)^4 = e^4 + 4e^3\left(\frac{1}{e}\right) + 6e^2\left(\frac{1}{e}\right)^2 + 4e\left(\frac{1}{e}\right)^3 + \left(\frac{1}{e}\right)^4 = e^4 + 4e^2 + 6 + \frac{4}{e^2} + \frac{1}{e^4}$$

(b)
$$\left(e - \frac{1}{e}\right)^4 = e^4 - 4e^3\left(\frac{1}{e}\right) + 6e^2\left(\frac{1}{e}\right)^2 - 4e\left(\frac{1}{e}\right)^3 + \left(\frac{1}{e}\right)^4 = e^4 - 4e^2 + 6 - \frac{4}{e^2} + \frac{1}{e^4}$$

Adding gives $2e^4 + 12 + \frac{2}{e^4}$

17. $\dots + 6 \times 2^2(ax)^2 + 4 \times 2(ax)^3 + (ax)^4 = \dots + 24a^2x^2 + 8a^3x^3 + a^4x^4$

18.
$$(3x + 2y)^4 = (3x)^4 + \binom{4}{1}(3x)^3(2y) + \binom{4}{2}(3x)^2(2y)^2 + \binom{4}{3}(3x)(2y)^3 + (2y)^4$$

$$= 81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$$

19. (i)
$$\binom{7}{3} = \frac{7!}{3!4!} = \frac{5 \times 6 \times 7}{1 \times 2 \times 3} = 35$$

(ii)
$$\binom{200}{2} = \frac{200!}{2! \times 198!} = \frac{199 \times 200}{2} = 199 \times 100 = 19900$$

20. (a)
$$\text{LHS} = \binom{5}{2} + \binom{5}{3} = \frac{5!}{2!3!} + \frac{5!}{3!2!} = 10 + 10 = 20, \text{ RHS} = \binom{6}{3} = \frac{6!}{3!3!} = 20$$

(b)
$$\text{LHS} = \binom{19}{9} + \binom{19}{10} = \frac{19!}{9!10!} + \frac{19!}{10!9!} = \frac{2 \times 19!}{9!10!} = \frac{2 \times 19!}{9!10!}$$

$$\text{RHS} = \binom{20}{10} = \frac{20!}{10!10!} = \frac{20 \times 19!}{10 \times 9!10!} = \frac{2 \times 19!}{9!10!}$$

21. $(a + b)^{12}$

Coefficient of a^5b^7 is
$$\binom{12}{5} = \binom{12}{7} = 792$$

$$22. (5a+b)^7 = \dots + \binom{7}{4} (5a)^3 (b)^4 + \dots = \frac{7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4} \times 5^3 \times (a^3 b^4) = 35 \times 5^3 \times a^3 b^4$$

So the coefficient is 4375

$$23. (a) \quad n = 10$$

$$(b) \quad a = p, b = 2q \text{ (or } a = 2q, b = p)$$

$$(c) \quad \binom{10}{5} p^5 (2q)^5$$

$$24. \text{ The required term is } \binom{10}{7} 2^{10-7} 3^7 \\ = 2099520$$

$$25. (a) \quad 12 \text{ terms}$$

$$(b) \quad (10\text{th term, } r = 9, \binom{11}{9} (x)^2 (2)^9)$$

$$\binom{11}{9} (x)^2 (2)^9 = 55 \times 2^9 x^2 = 28160x^2$$

$$26. \text{ Term involving } x^3 \text{ is } \binom{5}{3} (2)^2 (-x)^3 = -40x^3 \Rightarrow \text{The coefficient is } -40$$

$$27. \text{ Required term is } \binom{8}{5} (3x)^5 (-2)^3$$

Therefore the coefficient of x^5 is $56 \times 243 \times -8 = -108864$

$$28. \binom{8}{3} (2)^5 (-3x)^3$$

Term is $-48384x^3$

$$28. \text{ The coefficient of } x^3 \text{ is } \binom{8}{3} \left(\frac{-1}{2} \right)^3$$

The coefficient of x^3 is -7

30.

$$\text{Term in } x^3 = {}^6C_3 \times 2^3 \left(\frac{-3x}{2} \right)^3 \\ = 20 \times 8 \times \left(-\frac{27x^3}{8} \right) \\ = -540x^3$$

(Coefficient of $x^3 = -540$)

$$31. \binom{8}{5} \left(\frac{2}{3} x \right)^3 (-3)^5 = -4032x^3$$

$$32. a = \binom{10}{8} \times 2^2 = 45 \times 4 = 180$$

$$33. \binom{7}{2} 5^2 (2x^2)^5 = 16\,800 x^{10}$$

34. (a) 6 terms

$$(b) \text{ the fourth term is } \binom{5}{3} (-2)^3 (x^2)^2 = 10 (-8) x^4 = -80x^4 \quad \text{hence } A = -80$$

$$35. (a) \text{ 7 terms}$$

$$(b) \binom{6}{3} (x)^3 (-3x)^3$$

$$\binom{6}{3} = 20, (-3)^3 = -27$$

$$\text{Term is } -540x^{12}$$

$$36. \binom{5}{2} (3x^2)^3 \left(\frac{-2}{x}\right)^2 \binom{5}{2}, 27x^6, \frac{4}{x^2}; 10(3x^2)^3 \left(\frac{-2}{x}\right)^2 = 10 \times 27 \times 4 x^4$$

$$\text{term} = 1080x^4$$

$$37. (a) 10$$

$$(b) \binom{9}{6} (3x^2)^3 \left(-\frac{1}{x}\right)^6 = 84 \times 3^3 x^6 \frac{1}{x^6}$$

$$\text{constant} = 2268$$

$$38. \binom{9}{3} x^6 \left(\frac{-2}{x^2}\right)^3 = 84x^6 \left(\frac{-8}{x^6}\right) = -672$$

$$39. \binom{6}{3} (x^2)^3 \left(-\frac{2}{x^2}\right)^3 = -160$$

40.

$$\binom{5}{2} (3x)^3 \left(\frac{-2}{x}\right)^2$$

$$= 10(27x^3) \left(\frac{4}{x^2}\right)$$

$$= 1080x$$

$$\text{Coefficient is } 1080$$

$$41. \binom{10}{3} 2^7 (ax)^3 \left(\text{accept } \binom{10}{7}\right)$$

$$120 \times 2^7 a^3 = 414\,720$$

$$a^3 = 27 \Leftrightarrow a = 3$$

$$42. T_{r+1} = \binom{7}{r} x^{7-r} \left(\frac{1}{ax^2}\right)^r = \binom{7}{r} \left(\frac{1}{a}\right)^r x^{7-3r}$$

$$\text{Now, } \binom{7}{2} \frac{1}{a^2} = \frac{7}{3} \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$$

43. (a) $(x-2)^4 = x^4 + 4x^3(-2) + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4$
 $(x-2)^4 = x^4 - 8x^3 + 24x^2 - 32x + 16$
- (b) finding coefficients, $3 \times 24 (= 72)$, $4 \times (-8) (= -32)$
term is $40x^3$
44. (a) $2^4 + 4(2^3)x + 6(2^2)x^2 + 4(2)x^3 + x^4, (4 + 4x + x^2)(4 + 4x + x^2)$
 $(2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$
- (b) finding coefficients 24 and 1
term is $25x^2$
45. (a) $(2+x)^5 = 25 + 5(2)^4(x) + 10(2)^3x^2 + 10(2)^2x^3 + 5(2)x^4 + x^5$
 $= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
- (b) Let $x = 0.01$
 $\Rightarrow (2.01)^5 = 32 + 0.8 + 0.008 + 0.0004 + 0.00001 + 0.0000001$
 $= 32.8080401001$
- (c) Let $x = -0.01$
 $\Rightarrow (1.99)^5 = 32 - 0.8 + 0.008 - 0.0004 + 0.00001 - 0.0000001$
 $= 31.2079600999$

46.

METHOD 1

constant term: $\binom{5}{0}(-2x)^0 \binom{7}{0}x^0 = 1$

term in x : $\binom{7}{1}x + \binom{5}{1}(-2x) = -3x$

term in x^2 : $\binom{7}{2}x^2 + \binom{5}{2}(-2x)^2 + \binom{7}{1}x \binom{5}{1}(-2x) = -9x^2$

METHOD 2

$$\begin{aligned} (1-2x)^5(1+x)^7 &= \left(1 + 5(-2x) + \frac{5 \times 4(-2x)^2}{2!} + \dots\right) \left(1 + 7x + \frac{7 \times 6}{2}x^2 + \dots\right) \\ &= (1 - 10x + 40x^2 + \dots)(1 + 7x + 21x^2 + \dots) \\ &= 1 + 7x + 21x^2 - 10x - 70x^2 + 40x^2 + \dots \\ &= 1 - 3x - 9x^2 + \dots \end{aligned}$$

47. Given $(1+x)^5(1+ax)^6 = 1 + bx + 10x^2 + \dots + a^6x^{11}$
 $\Leftrightarrow (1 + 5x + 10x^2 + \dots)(1 + 6ax + 15a^2x^2 + \dots) = 1 + bx + 10x^2 + \dots + a^6x^{11}$
 $\Leftrightarrow 1 + (6a + 5)x + (15a^2 + 30a + 10)x^2 + \dots = 1 + bx + 10x^2 + \dots + a^6x^{11}$
 $\Leftrightarrow$

$$6a + 5 = b \quad (1)$$

$$15a^2 + 30a + 10 = 10 \quad (2)$$

$$(2) \Rightarrow 15a^2 + 30a = 0$$

$$\Leftrightarrow 15a(a + 2) = 0$$

$$\Rightarrow a = -2$$

Substitute into (1) $b = -7$

Note: $a \neq 0$ since $a \in \mathbb{Z}^*$

48.

(a) evidence of expanding

e.g. $2^4 + 4(2^3)x + 6(2^2)x^2 + 4(2)x^3 + x^4, (4+4x+x^2)(4+4x+x^2)$

$$(2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$$

(b) finding coefficients 24 and 1

term is $25x^2$

49 The terms of $(2+x)^4$ involve $x^4, x^3, x^2, x^1, x^0 = 1$

The terms of $(1+x^2)^5$ involve $x^{10}, x^8, x^6, x^4, x^2, x^0 = 1$

The combinations that give x^3 are shown above.

The first combination is $\binom{4}{3} 2^1 x^3 \times \binom{5}{5} 1^5 (x^2)^0 = 8x^3 \times 1 = 8x^3$

The second combination is $\binom{4}{1} 2^3 x^1 \times \binom{5}{4} 1^4 (x^2)^1 = 32x \times 5x^2 = 160x^3$

Since $8x^3 + 160x^3 = 168x^3$, the coefficient of x^3 is 168.

50. The terms of $(2+x)^4$ involve $x^4, x^3, x^2, x^1, x^0 = 1$

The terms of $\left(2x + \frac{1}{x}\right)^5$ involve $x^5, x^3, x^1, x^{-1}, x^{-3}, x^{-5}$

The combinations that give constant term are shown above.

The first combination is $\binom{4}{3} 2^1 x^3 \times \binom{5}{4} (2x)^1 \left(\frac{1}{x}\right)^4 = 8x^3 \times \frac{10}{x^3} = 80$

The second combination is $\binom{4}{1} 2^3 x^1 \times \binom{5}{3} (2x)^2 \left(\frac{1}{x}\right)^3 = 32x \times \frac{40}{x} = 1280$

Hence the constant term is $80 + 1280 = 1360$.

51. (a) $2x^2 + x - 3 = (2x + 3)(x - 1)$

Note: Accept $2 \overline{) x + \frac{3}{2}} \overline{) (x - 1)}$. Either of these may be seen in (b) and A1 should be awarded.

(b) **EITHER**

$$(2x^2 + x - 3)^8 = (2x + 3)^8 (x - 1)^8$$

$$= (3^8 + 8(3^7)(2x) + \dots)((-1)^8 + 8(-1)^7(x) + \dots)$$

$$\text{coefficient of } x = 3^8 \times 8 \times (-1)^7 + 3^7 \times 8 \times 2 \times (-1)^8 = -17\,496$$

OR

$$(2x^2 + x - 3)^8 = (3 - (x - 2x^2))^8 = 3^8 + 8(-(x - 2x^2))(3^7) + \dots$$

$$\text{coefficient of } x = 8 \times (-1) \times 3^7 = -17\,496$$

52. (b) 61236 and 30618

53. We expand up to the third term

$$(1+ax)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}(ax) + \binom{n}{2}1^{n-2}(ax)^2 + \dots$$

$$= 1 + nax + \frac{n(n-1)}{2}a^2x^2 + \dots$$

Hence $na = 24$ and $\frac{n(n-1)}{2}a^2 = 252 \Leftrightarrow (n^2 - n)a^2 = 504$

The solution if the system is $a = 3$ and $n = 8$.

54. (a) coefficient of x^3 is $\binom{n}{3}\binom{1}{2}^3 = 70$

$$\frac{n!}{3!(n-3)!} \times \frac{1}{8} = 70$$

$$\Rightarrow \frac{n(n-1)(n-2)}{48} = 70$$

$$n = 16$$

(b) $\binom{16}{2}\binom{1}{2}^2 = 30$

B. Exam style questions (LONG)

55. (a) $(1+1)^4 = 2^4 = 1 + \binom{4}{1}(1) + \binom{4}{2}(1^2) + \binom{4}{3}(1^3) + 1^4 \Rightarrow \binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 16 - 2 = 14$

(b) For $x = -1$

$$0 = 1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + 1 \Rightarrow \binom{4}{1} - \binom{4}{2} + \binom{4}{3} = 2$$

(c) The expansion of $(1+x)^9$ gives

$$(1+1)^9 = 1 + \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \dots + \binom{9}{8} + 1$$

$$\Rightarrow \binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \dots + \binom{9}{8} = 2^9 - 2 = 510$$

(d) For $x = -1$

$$1 - \binom{9}{1} + \binom{9}{2} - \binom{9}{3} + \binom{9}{4} - \binom{9}{5} + \binom{9}{6} - \binom{9}{7} + \binom{9}{8} - 1 = 0$$

$$\Rightarrow \binom{9}{1} + \binom{9}{3} + \binom{9}{5} + \binom{9}{7} = \binom{9}{2} + \binom{9}{4} + \binom{9}{6} + \binom{9}{8}$$

56. (a) $(a+b)^2 = a^2 + b^2 + 2ab \Rightarrow a^2 + b^2 = S^2 - 2P$

(b) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$

$$\Rightarrow a^3 + b^3 = S^3 - 3SP$$

(c) $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = a^4 + b^4 + 4ab(a^2 + b^2) + 6a^2b^2$

$$\Rightarrow a^4 + b^4 = S^4 - 4P(a^2 + b^2) - 6P^2 \Rightarrow a^4 + b^4 = S^4 - 4P(S^2 - 2P) - 6P^2$$

$$\Rightarrow a^4 + b^4 = S^4 - 4S^2P + 2P^2$$