

[MAA 1.2-1.3] ARITHMETIC SEQUENCES

SOLUTIONS

O. Practice questions

SEQUENCES (IN GENERAL)

1. (a) $u_1 = 2$, $S_1 = 2$.
 (b) $u_1 = 5$, $S_2 = 7$.
 (c) $S_5 = 2 + 5 + 10 + 3 + 7 = 27$.
2. (a) $u_2 = 2$, $u_{20} = 20$.
 (b) $S_5 = 1 + 2 + 3 + 4 + 5 = 15$.
 (c) $u_n = n$.
3. (a) 10, 20, 30
 (b) $u_{10} = 100$
 (c) $S_1 = 10$, $S_2 = 30$, $S_3 = 60$
4. (a) 10, 40, 90
 (b) $u_{10} = 1000$
 (c) $S_1 = 10$, $S_2 = 50$, $S_3 = 140$
5. (a) 10, 100, 1000
 (b) $u_{10} = 10,000,000,000$
 (c) $S_1 = 10$, $S_2 = 110$, $S_3 = 1110$
6. (a) 10, 20, 30
 (b) $S_1 = 10$, $S_2 = 30$, $S_3 = 60$
7. (a) 10, 30, 70
 (b) $S_1 = 10$, $S_2 = 40$, $S_3 = 110$
8. (a) $\sum_{r=1}^3 (2r) = 2 + 4 + 6 = 12$
 (b) $\sum_{r=1}^3 r^2 = 1 + 4 + 9 = 14$
 (c) $\sum_{r=1}^3 2^r = 2 + 4 + 8 = 14$
9. (a) $A = \sum_{r=1}^{10} (2r^2 + 1) = 780$, $B = \sum_{r=1}^{20} (2r^2 + 1) = 5760$,
 $C = \sum_{r=11}^{20} (2r^2 + 1) = 4980$
 (b) $A + C = B$ or $C = B - A$

ARITHMETIC SEQUENCES

10. $u_1 = 11, d = 4.$

(a) $u_{101} = 11 + 100 \times 4 = 411, S_{101} = \frac{101}{2}(11 + 411) = 21311$

(b) $S_{20} = \frac{20}{2}(2 \times 11 + 19 \times 4) = 980$

(c) $u_n = 11 + (n-1)4 = 11 + 4n - 4 = 4n + 7$

(d) $u_n = 51 \Leftrightarrow 4n + 7 = 51 \Leftrightarrow 4n = 44 \Leftrightarrow n = 11$

11. (a) $d=8$ (b) 75 (c) 385 (d) $n = 21$

12. (a) $u_1 = 10, d = 5$ (b) $u_n = 5n + 5$ (c) $S_n = \frac{n}{2}(5n + 15)$ (d) $u_n = 105$ and $S_{20} = 1150$

13. (a) $u_1 = 10, u_2 = 15$ (b) $d = 5$ (c) $S_3 = 45, S_4 = 70$ (d) $S_n = \frac{n}{2}(5n + 15)$

14. (a) (i) 15 (ii) 40400 (iii) 40385 (b) 30200

15. (a) 23 (b) 99 (c) 1265

A. Exam style questions (SHORT)

16. (a) $u_1 = 7, d = 2.5$

$$u_{41} = u_1 + (n-1)d = 7 + 40 \times 2.5 = 107$$

(b) $S_{101} = \frac{101}{2}[2(7) + 100 \times 2.5] = \frac{101(264)}{2} = 13332$

17. $S_5 = \frac{5}{2}\{2 + 32\} = 85$

OR

$$a = 2, a + 4d = 32 \Rightarrow 4d = 30 \Rightarrow d = 7.5$$

$$S_5 = \frac{5}{2}(4 + 4(7.5)) = \frac{5}{2}(4 + 30) = 85$$

18. (a) $u_3 = u_1 + 2d \Leftrightarrow 8 = 2 + 2d \Leftrightarrow d = 3$

(b) $u_{20} = 2 + 19 \times 3 = 59$

(c) $S_{20} = \frac{20}{2}(2 + 59) = 610$ **OR** $S_{20} = \frac{20}{2}(2 \times 2 + 19 \times 3) = 610$

19. $u_4 = 40 \Leftrightarrow 5 + 3d = 40 \Leftrightarrow d = \frac{35}{3}$

$$u_2 = 5 + \frac{35}{3} = \frac{50}{3}$$

20. Arithmetic sequence

$$u_1 = 200, d = 30$$

(a) Distance in final week = $200 + 51 \times 30 = 1730$ m

(b) Total distance = $\frac{52}{2}[2 \times 200 + 51 \times 30] = 50180$ m

21. (a) Arithmetic sequence
 $u_1 = 15$ $d = 2$ $n = 20$
 $u_{20} = 15 + 19 \times 2 = 53$ (that is, 53 seats in the 20th row)
- (b) $S_{20} = \frac{20}{2} (2 \times 15 + 19 \times 2)$ (or $\frac{20}{2} (15 + 53)$)
 $= 680$ (that is, 680 seats in total)
22. (a) $u_1 = 1000$, $u_n = 1000 + (n - 1)250 = 10\,000$
 $n = 37$.
She runs 10 km on the 37th day.
- (b) $S_{37} = \frac{37}{2} (1000 + 10\,000) = 203\,500$. She has run a total of 203.5 km
23. (a) $T_1 = 100$ $d = 25$
 $T_{17} = 100 + 16 \times 25 = \500
- (b) $S_{17} = \frac{17}{2} (100 + 500) = \5100
24. (a) $45000 + 4 \times 1750 = 52000$ USD
- (b) $\frac{10}{2} (2(45000) + 9 \times (1750)) = 528750$ USD (Accept 529000)
25. (a) $20 = u_1 + 3d$
 $32 = u_1 + 7d$
 $d = 3$ (and $u_1 = 11$)
- (b) $\frac{10}{2} (2 \times 11 + 9 \times 3) = 245$
26. (a) $u_{21} = 24 + 20 \times (16) = 344$
- (b) $S_{31} = \frac{31}{2} [2(24) + (30)(16)] = 8184$
27. (a) $u_{20} = u_1 + 19d \Leftrightarrow 64 = 7 + 19d$, $d = \frac{64 - 7}{19} = 3$
- (b) $u_n = 3709 \Leftrightarrow 3709 = 7 + 3(n - 1) \Leftrightarrow n = 1235$
28. $u_2 = u_1 + d = 7$ and $S_4 = \frac{4}{2} (2u_1 + 3d) = 12$
 $\Rightarrow u_1 + d = 7$
 $4u_1 + 6d = 12 \Rightarrow u_1 = 15, d = -8$
29. $u_2 = u_1 + d \Rightarrow u_1 + d = 7$
 $S_5 = \frac{5}{2} (2u_1 + 4d) = 50 \Rightarrow 2u_1 + 4d = 20$
 $d = 3$
30. (a) $u_{27} = 263 \Leftrightarrow 263 = u_1 + 26 \times 11 \Leftrightarrow u_1 = -23$
- (b) (i) $516 = -23 + (n - 1) \times 11 \Leftrightarrow n = 50$
- (ii) $S_{50} = \frac{50(-23 + 516)}{2} = 12325$ OR $S_{50} = \frac{50(2 \times (-23) + 49 \times 11)}{2} = 12325$

31. (a) $d = 3$
 $u_{101} = 2 + 100 \times 3 = 302$
- (b) $152 = 2 + (n - 1) \times 3 \Leftrightarrow 150 = (n - 1) \times 3 \Leftrightarrow 50 = n - 1 \Leftrightarrow n = 51$
32. (a) $S_n = \frac{n}{2} (2 \times 2 + 3(n - 1)) = \frac{n}{2} (3n + 1)$
- (b) $\frac{n}{2} (3n + 1) = 1365 \Rightarrow 3n^2 + n - 2730 = 0$
 $\Rightarrow n = 30$ or $n = \frac{-91}{3}$, hence $n = 30$
33. (a) $u_4 = u_1 + 3d \Leftrightarrow 16 = -2 + 3d \Leftrightarrow d = 6$
- (b) $u_n = u_1 + (n - 1)6 \Leftrightarrow 11998 = -2 + (n - 1)6 \Leftrightarrow n = \frac{11998 + 2}{6} + 1 = 2001$
34. (a) (i) $-37 = u_1 + 20d$
 $-3 = u_1 + 3d$
 $-34 = 17d \Leftrightarrow d = -2$
- (ii) $-3 = u_1 - 6 \Rightarrow u_1 = 3$
- (b) $u_{10} = 3 + 9 \times -2 = -15$
 $S_{10} = \frac{10}{2} (3 + (-15)) = -60$
35. (a) $\frac{20}{2} \{2(-7) + 19d\} = 620 \Leftrightarrow d = 4$
- (b) $u_{78} = -7 + 77(4) = 301$
36. (a) 3, 6, 9
- (b) (i) sum of an AP $\sum_{n=1}^{20} 3n = \frac{20}{2} 2 \times 3 + (20 - 1) \times 3 = 630$
- (ii) **METHOD 1**
 $\sum_{n=1}^{100} 3n = \frac{100}{2} (2 \times 3 + 99 \times 3) = 15150$
 $\sum_{n=21}^{100} 3n = 15150 - 630 = 14520$
- METHOD 2**
first term is 63, the number of terms is 80
 $\sum_{n=21}^{100} 3n = \frac{80}{2} (63 + 300) = 14520$
37. (a) $u_1 = 1, u_2 = -1, u_3 = -3$
- (b) $S_{20} = \frac{20}{2} (2 \times 1 + 19 \times -2) (= 10(2 - 38)) = -360$

38. (a) $d = 2$
 (b) (i) $5 + 2n = 115 \Rightarrow n = 55$
 (ii) $u_1 = 7$
 $S_{55} = \frac{55}{2}(7 + 115) = 3355$ OR $S_{55} = \frac{55}{2}(2(7) + 54(2)) = 3355$ OR $\sum_{k=1}^{55} (5 + 2k) = 3355$

39. (a) common difference is 6
 (b) $u_n = 1353 \Leftrightarrow 1353 = 3 + (n - 1)6 \Leftrightarrow n = 226$
 (c) $S_{226} = \frac{226(3 + 1353)}{2}$ OR $S_{226} = \frac{226}{2}(2 \times 3 + 225 \times 6)$
 $S_{226} = 153\,228$ (153 000 is also accepted)

40. **METHOD 1**

substituting into formula for S_{40} : $1900 = \frac{40(u_1 + 106)}{2} \Leftrightarrow u_1 = -11$

substituting into formula for u_{40} : $106 = -11 + 39d \Leftrightarrow d = 3$

METHOD 2

substituting into formulas for S_{40} and u_{40}

$$20(2u_1 + 39d) = 1900$$

$$u_1 + 39d = 106$$

Solution: $u_1 = -11, d = 3$

41. (a) $u_{20} = u_1 + 19d \Leftrightarrow 64 = 7 + 19d, \Leftrightarrow d = 3$
 (b) $3709 = 7 + 3(n - 1) \Leftrightarrow 3709 = 3n + 4 \Leftrightarrow n = 1235$

42. (a) $n=21$ (b) 1575

43. (a) $u_1 = 5$ and $d = 8$
 $u_n = u_1 + (n - 1)d \Rightarrow u_n = 8n - 3$
 (b) $8n - 3 < 400 \Rightarrow 8n < 403$
 $n < 50.375$
 Therefore, there are 50 terms less than 400.

44. (a) $u_1 = S_1 = 7$
 (b) $u_2 = S_2 - u_1 = 18 - 7 = 11$
 $d = 11 - 7 = 4$
 (c) $u_4 = u_1 + (n - 1)d = 7 + 3(4)$
 $u_4 = 19$

45. Arithmetic sequence $d = 3$
 $n = 1250$
 $S = \frac{1250}{2}(3 + 3750) = 2\,345\,625$ OR $S = \frac{1250}{2}(6 + 1249 \times 3) = 2\,345\,625$

46. $81 = \frac{n}{2}(1.5 + 7.5) \Rightarrow n = 18$
 $1.5 + 17d = 7.5 \Rightarrow d = \frac{6}{17}$

47. $17 + (n-1)10 = 417 \Leftrightarrow 10(n-1) = 400$ so $n = 41$

$$S_{41} = \frac{41}{2}(2(17) + 40(10)) = 41(17 + 200) = 8897 \quad \text{OR} \quad S_{41} = \frac{41}{2}(17 + 417) = \frac{41}{2}(434) = 8897$$

48. (a) $n = 51$, $S_{51} = 4182$

(b) $\sum_{r=1}^{51} (3r + 4) = 4182$ (it is same sum!)

49. (a) $u_n = 5 + 4(n-1) = 4n + 1$

(b) $\sum_{r=1}^{21} (4r + 1)$

50. Arithmetic progression: 85, 78, 71, ...

$$u_1 = 85, d = -7$$

$$u_n = 85 - 7(n-1) = 92 - 7n \quad u_n > 0 \Rightarrow n \leq 13.$$

$$S_{13} = \frac{13}{2}(u_1 + u_{13}) = \frac{13}{2}(85 + 1) = 559$$

51. 4th term = $a + 3d$

$$8^{\text{th}} \text{ term} = a + 7d$$

$$20^{\text{th}} \text{ term} = a + 19d$$

We have two relations:

$$a + 7d = 2(a + 3d) \Leftrightarrow a + 7d = 2a + 6d \Leftrightarrow a = d$$

$$a + 19d = 4000$$

The solution is $d = 200$ (and $a = 200$)

52. $u_1 = -6$ and $d = 7$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \Rightarrow S_n = \frac{n}{2}(7n - 19)$$

Solving $S_n > 10000 \Rightarrow n > 54.8$, The least number of terms is 55

53. (a) $(2a + 4) - (a + 3) = (a + 9) - (2a + 4) \Leftrightarrow 2a = 4 \Leftrightarrow a = 2$

(b) the terms are 5, 9, 11 which are indeed in AS with $d = 2$

54. $(2a + b + 7) - (a - b) = (a - b) - 2$

$$(a - 3b) - (2a + b + 7) = (2a + b + 7) - (a - b)$$

$$0 = 2a + 6b + 14$$

$$b = -3 \quad a = 2$$

55. Let a be the first term and d be the common difference of the arithmetic sequence.

$$\text{Then } \frac{a+4d}{a+11d} = \frac{6}{13}$$

$$\text{So } 13a + 52d = 6a + 66d \Rightarrow 7a = 14d \Rightarrow a = 2d.$$

Since each term is positive, both a and d are positive.

We are given $a(a+2d) = 32$, setting $a = 2d$, we get $2d(2d+2d) = 8d^2 = 32$.

$$\Rightarrow d = \pm 2.$$

Hence, $d = 2$ and $a = 4$ and sum to 100 terms of this sequence is

$$\frac{100}{2} \{(2)(4) + (100-1)2\} = 10\,300$$

56. (a) $S_1=1, S_2=8$, hence $u_1=1, u_2=7$

$$(b) \quad u_n = S_n - S_{n-1} = [3n^2 - 2n] - [3(n-1)^2 - 2(n-1)] = 6n - 5$$

OR

$$u_n = 1 + 6(n-1) = 6n - 5$$

57. (a) $S_n = 2n^2 - n$

$$n=1 \Rightarrow S_1 = u_1 = 2 - 1 = 1$$

$$n=2 \Rightarrow S_2 = u_1 + u_2 = 8 - 2 = 6 \Rightarrow u_2 = 5$$

$$n=3 \Rightarrow S_3 = u_1 + u_2 + u_3 = 18 - 3 = 15 \Rightarrow u_3 = 9$$

$$(b) \quad u_n = S_n - S_{n-1}$$

$$\Rightarrow u_n = 2n^2 - n - (2(n-1)^2 - (n-1))$$

$$\Rightarrow u_n = 2n^2 - n - (2n^2 - 4n + 2 - n + 1)$$

$$\Rightarrow u_n = 4n - 3$$

58. (a) $S_4 = 68 \quad S_5 = 105 \quad u_5 = 37$ (b) $u_n = 8n - 3$ (c) $u_n - u_{n-1} = 8$

B. Exam style questions (LONG)

59. (a) 24 (b) 97 (c) 1224 (d) 21

60. (a) $u_1 = 1, n = 20, u_{20} = 20$ ($u_1 = 1, n = 20, d = 1$)

$$S_{20} = \frac{(1+20)20}{2} \quad (\text{or } S = \frac{20}{2}(2 \times 1 + 19 \times 1)) = 210$$

- (b) Let there be n cans in bottom row

$$S_n = 3240 \Leftrightarrow \frac{(1+n)n}{2} = 3240 \Leftrightarrow n^2 + n - 6480 = 0 \Leftrightarrow n = 80 \text{ or } n = -81$$

So $n = 80$

$$(c) \quad (i) \quad S = \frac{(1+n)n}{2} \Leftrightarrow 2S = n^2 + n \Leftrightarrow n^2 + n - 2S = 0$$

- (ii) **METHOD 1**

$$\text{Substituting } S = 2100: n^2 + n - 4200 = 0 \Leftrightarrow n = 64.3, n = -65.3$$

n must be a (positive) integer, this equation does not have integer solutions.

METHOD 2

Trial and error: $S_{64} = 2080, S_{65} = 2145$ integer not possible here

61. (a) $u_1 = 1, d = 3 \quad u_{11} = 31$
 (b) $S_n = \frac{n}{2}(2 + (n-1) \times 3) = \frac{n}{2}(2 + 3n - 3) = \frac{n}{2}(3n - 1).$
 (c) (i) $\frac{100}{2}(3 \times 100 - 1) = 14950$
 (d) (i) $\frac{n}{2}(3n - 1) = 477 \Leftrightarrow 3n^2 - n = 954 \Leftrightarrow 3n^2 - n - 954 = 0$
 (ii) 18

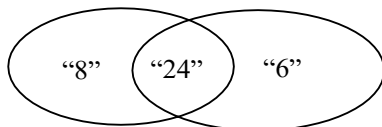
62. (a) $(5k - 2) - (2k + 3) = (10k - 15) - (5k - 2)$
 $\Leftrightarrow 5k - 2 - 2k - 3 = 10k - 15 - 5k + 2$
 $\Leftrightarrow 3k - 5 = 5k - 13 \Leftrightarrow -2k = -8 \text{ or } 2k = 8$
 $\Leftrightarrow k = 4$
 (b) 11, 18, 25
 (c) 7
 (d) $U_{20} = 11 + 19 \times 7 = 144$
 (e) $S_{15} = \frac{15}{2}(2 \times 11 + 14 \times 7) = 900$

63. (a) (i) 6, 9, 12 (ii) 3
 (b) $x^2 - (x^2 - 3) = 4x - x^2 \Leftrightarrow x^2 - 4x^2 + 3 = 0 \Leftrightarrow x = 1 \text{ or } x = 3$
 So the other value is 1.
 (d) (i) -2, 1, 4
 (ii) 3
 (iii) $-2 + 1 + 4 + 7 = 10$

64. The multiples of 8 form an arithmetic sequence and have the form $8n$

- (a) $\frac{900}{8} = 112.5$, hence $\sum_{n=1}^{112} 8n = 50624$
 (b) $\frac{100}{8} = 12.5$ hence $\sum_{x=13}^{112} 8n = 50000$

For the remaining questions it helps to draw a Venn diagram for the multiples of 8 and 6:



- (c) The common multiples are the multiples of 24. $\frac{900}{24} = 37.5$, hence $\sum_{n=1}^{37} 24n = 16872$
 (d) $\sum_{x=13}^{112} 8n - \sum_{x=1}^{37} 24n = 50624 - 16872 = 33752$
 (e) $\frac{900}{6} = 150$ hence the sum of the multiples of 6 is $\sum_{n=1}^{149} 6n = 67050$

The sum of all multiples either of 8 or of 6 is

$$\sum_{x=13}^{112} 8n + \sum_{n=1}^{149} 6n - \sum_{x=1}^{37} 24n = 50624 + 67050 - 16872 = 100802$$

65. The sizes form an AS with $u_1 = 1$ and $d = 3$

- (a) (i) the 20th group contains $u_{20} = 1 + 19 \times 3 = 58$ numbers
 (ii) the last term of the 20th group is $S_{20} = 590$
- (b) (i) the nth group contains $u_n = 1 + (n - 1)3 = 3n - 2$ numbers
 (ii) the last term of the nth group is $S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1 + 3n - 2) = \frac{n(3n - 1)}{2}$
- (c) the first term of the 20th group is $S_{19} + 1 = 533$
- (d) the sum of the terms in the 20th group is $\frac{58}{2}(533 + 590) = 32567$