

[MAA 1.11-1.12] COMPLEX NUMBERS (CARTESIAN FORM)

SOLUTIONS

CARTESIAN FORM

O. Practice questions

1. (a) $\Delta = -16$
 (b) $z = \frac{8 \pm 4i}{2} = 4 \pm 2i$
 (c) $(z - 4 - 2i)(z - 4 - 2i) = (z - 4)^2 - 4i^2 = (z - 4)^2 + 4$
2. (a) $\Delta = -144$
 $z = \frac{8 \pm 12i}{8} = \frac{8}{8} \pm \frac{12}{8}i = 1 \pm \frac{3}{2}i$
 (b) (i) $S = 1 + \frac{3}{2}i + 1 - \frac{3}{2}i = 2$ and $-\frac{b}{a} = \frac{8}{4} = 2$
 (ii) $S = \left(1 + \frac{3}{2}i\right) + \left(1 - \frac{3}{2}i\right) = 1^2 - \left(\frac{3}{2}i\right)^2 = 1 + \frac{9}{4} = \frac{13}{4}$ and $\frac{c}{a} = \frac{13}{4}$
3. (a) (i) $z_1 + z_2 = 13 + 9i$,
 (ii) $z_2 - z_1 = 7 + i$
 (iii) $z_1 z_2 = (3 + 4i)(10 + 5i) = 30 + 15i + 40i - 20 = 10 + 55i$
 (iv) $\frac{z_2}{z_1} = \frac{10 + 5i}{3 + 4i} = \frac{10 + 5i}{3 + 4i} \times \frac{3 - 4i}{3 - 4i} = \frac{50 - 25i}{25} = 2 - i$
 (b) (i) $z_1^2 = (3 + 4i)^2 = 9 + 24i - 16 = -7 + 24i$
 (ii) $z_1^3 = (3 + 4i)^3 = 3^3 + 3 \times 3^2 \times 4i + 3 \times 3 \times (4i)^2 + (4i)^3$
 $= 27 + 108i - 144 - 64i = -117 + 44i$
 (OR $(3 + 4i)^2 (3 + 4i) = (-7 + 24i) (3 + 4i) = -117 + 44i$)
 (c) (i) $|z_1| = 5$,
 (ii) $|z_2| = \sqrt{125} = 5\sqrt{5}$,
 (iii) $|z_2 - z_1| = \sqrt{50} = 5\sqrt{2}$
4. (a) $(1 - i\sqrt{3})^2 = 1^2 - 2\sqrt{3}i + (\sqrt{3}i)^2 = -2 - 2\sqrt{3}i$
 (b) $(1 - i\sqrt{3})^3 = (1 - i\sqrt{3})^2 (1 - i\sqrt{3}) = (-2 - 2i\sqrt{3})(1 - i\sqrt{3}) = -2 - 6 - 2i\sqrt{3} + 2i\sqrt{3} = -8$
5. (a) $a - 2 = 7 \Leftrightarrow a = 9$
 $b - 1 = 3 \Leftrightarrow b = 4$
 (b) $c - 2 = 0 \Leftrightarrow c = 2$
 $d - 1 = 0 \Leftrightarrow d = 1$
6. The final result is $3 + i$

A. Exam style questions (SHORT)

$$7. \quad z = 1 + \frac{i(i + \sqrt{3})}{(i - \sqrt{3})(i + \sqrt{3})} = 1 + \frac{i(i + \sqrt{3})}{-4} = \frac{-5 + i\sqrt{3}}{-4} = \frac{5}{4} - \frac{i\sqrt{3}}{4}$$

8.

$$(1 - \sqrt{3}i)(1 - \sqrt{3}i) = 1 - 2\sqrt{3}i - 3 = (-2 - 2\sqrt{3}i)$$

$$(-2 - 2\sqrt{3}i)(1 - \sqrt{3}i) = -8$$

$$\therefore \frac{1}{(1 - \sqrt{3}i)^3} = -\frac{1}{8}$$

OR

Attempt at Binomial expansion

$$(1 - \sqrt{3}i)^3 = 1 + 3(-\sqrt{3}i) + 3(-\sqrt{3}i)^2 + (-\sqrt{3}i)^3$$

$$= 1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i$$

$$= -8$$

$$\therefore \frac{1}{(1 - \sqrt{3}i)^3} = -\frac{1}{8}$$

$$9. \quad z = \frac{2}{(1-i)} \cdot \frac{(1+i)}{(1+i)} + 1 - 4i = 1 + i + 1 - 4i = 2 - 3i$$

$$z^2 = (2 - 3i)^2 = -5 - 12i \quad (\text{or } x = -5, y = -12)$$

$$10. \quad 2(p + iq) = q - ip - 2(1 - i)$$

$$2p = q - 2$$

$$2q = -p + 2$$

$$p = -0.4, q = 1.2$$

$$11. \quad (a + i)(2 - bi) = 7 - i \Rightarrow 2a - abi + 2i - bi^2 = 7 - i$$

$$\Rightarrow 2a - abi + 2i + b = 7 - i$$

Equating real and imaginary parts

$$2a + b = 7$$

$$2 - ab = -1 \Rightarrow ab = 3$$

$$\text{Substitution } \Rightarrow 2a^2 - 7a + 3 = 0 \Rightarrow a = 3 \text{ or } a = 1/2 \text{ (rejected).}$$

Therefore, $a = 3$ and $b = 1$

$$12. \quad (a + bi)(2 - i) = (5 - i)$$

$$(a + bi) = \frac{(5 - i)}{(2 - i)} = \frac{11}{5} + \frac{3}{5}i \quad (\text{using a GDC}).$$

$$\text{Therefore } a = \frac{11}{5}, b = \frac{3}{5}$$

$$\text{OR } a + bi = \frac{(5 - i)}{(2 - i)} \times \frac{(2 + i)}{(2 + i)} = \frac{10 + 5i - 2i + 1}{4 + 1} = \frac{11 + 3i}{5} \quad \text{Therefore } a = \frac{11}{5}, b = \frac{3}{5}$$

$$\text{OR } (a + bi)(2 - i) = (5 - i)$$

$$(2a + b) + (2b - a)i = (5 - i)$$

$$2a + b = 5$$

$$-a + 2b = -1$$

$$\text{Therefore } a = \frac{11}{5}, b = \frac{3}{5}$$

13. Let $z = x + iy$
 $(1 - i)(x + iy) = 1 - 3i$
 $x + iy - i(x - y) = 1 - 3i$
 $\Rightarrow x = 2, y = -1$
 $x - y = 3$

OR

$$(1 - i)z = 1 - 3i \Leftrightarrow z = \frac{1 - 3i}{1 - i} \Leftrightarrow z = \frac{1 - 3i}{1 - i} \times \frac{1 + i}{1 + i} \Leftrightarrow z = 2 - i$$

14. $i(z + 2) = 1 - 2z \Rightarrow (2 + i)z = 1 - 2i$
 $\Rightarrow z = \frac{1 - 2i}{2 + i} = \frac{1 - 2i}{2 + i} \times \frac{2 - i}{2 - i} = \frac{-5i}{5} = -i. \quad (a = 0, b = -1)$

15. **METHOD 1**

By rationalizing we obtain

$$\frac{a}{2} + \frac{b}{5} = 3 \text{ and } -\frac{a}{2} + \frac{2b}{5} = 0$$

Solving gives $a = 4, b = 5$.

METHOD 2

$$\frac{a}{1+i} + \frac{b}{1-2i} = 3$$

$$a(1-2i) + b(1+i) = 3(1-2i)(1+i)$$

$$= 9 - 3i$$

$$\text{Re}(z): a + b = 9$$

$$\text{Im}(z): -2a + b = -3$$

$$3a = 12$$

$$a = 4, b = 5$$

16.

$$\frac{z(3-4i)}{(3+4i)(3-4i)} - \frac{5i(z-1)}{-5 \times 5i \times i} = \frac{5(3+4i)}{(3-4i)(3+4i)}$$

$$z(3-4i) - 5i(z-1) = 15 + 20i$$

$$z(3-4i-5i) = 15 + 20i - 5i$$

$$z = \frac{15+15i}{3-9i}$$

$$z = -1 + 2i$$

17. (a) $4+i$ (b) $3+i$

18. Let $z = a + bi$, so $z^* = a - bi$

$$|z|^2 = a^2 + b^2 = 20$$

$$\frac{25}{a+bi} - \frac{15}{a-bi} = 1-8i \Rightarrow \frac{25(a-bi) - 15(a+bi)}{a^2 + b^2} = 1-8i$$

$$\frac{10a}{20} = 1 \Rightarrow a = 2 \quad \text{and} \quad -\frac{40b}{20} = -8 \Rightarrow b = 4$$

Therefore, $z = 2 + 4i$

19.

Solving simultaneously

$$2z_1 + 3z_2 = 7$$

$$2z_1 + 2iz_2 = 8 + 8i$$

$$z_2(2i - 3) = 1 + 8i$$

$$z_2 = \frac{1 + 8i}{2i - 3} = 1 - 2i$$

$$z_1 = \frac{7 - 3(1 - 2i)}{2} \text{ or } 4 + 4i - i(1 - 2i) \\ = 2 + 3i$$

20.

METHOD 1

Substituting $z = x + iy$ to obtain $w = \frac{x + yi}{(x + yi)^2 + 1}$

$$w = \frac{x + yi}{x^2 - y^2 + 1 + 2xyi}$$

Use of $(x^2 - y^2 + 1 - 2xyi)$ to make the denominator real.

$$= \frac{(x + yi)(x^2 - y^2 + 1 - 2xyi)}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$\text{Im } w = \frac{y(x^2 - y^2 + 1) - 2x^2y}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$= \frac{y(1 - x^2 - y^2)}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$\text{Im } w = 0 \Rightarrow 1 - x^2 - y^2 = 0 \text{ i.e. } |z| = 1 \text{ as } y \neq 0$$

METHOD 2

$$w(z^2 + 1) = z$$

$$w(x^2 - y^2 + 1 + 2ixy) = x + yi$$

Equating real and imaginary parts

$$w(x^2 - y^2 + 1) = x \text{ and } 2wx = 1, y \neq 0$$

$$\text{Substituting } w = \frac{1}{2x} \text{ to give } \frac{x}{2} - \frac{y^2}{2x} + \frac{1}{2x} = x$$

$$-\frac{1}{2x}(y^2 - 1) = \frac{x}{2} \text{ or equivalent}$$

$$x^2 + y^2 = 1, \text{ i.e. } |z| = 1 \text{ as } y \neq 0$$

21. Let $z = x + iy$, $x, y \in \mathbb{R}$.

$$\text{Then, } |z + 16|^2 = 16|z + 1|^2$$

$$\Rightarrow (x + 16)^2 + y^2 = 16\{(x + 1)^2 + y^2\}$$

$$\Rightarrow x^2 + 32x + 256 + y^2 = 16x^2 + 32x + 16 + 16y^2$$

$$\Rightarrow 15x^2 + 15y^2 = 240$$

$$\Rightarrow x^2 + y^2 = 16$$

Therefore, $|z| = 4$.

POLYNOMIALS

O. Practice questions

22. (a) $f(1) = 1 - 3 + 7 - 5 = 0$
 (b) $z = 1 + 2i, z = 1 - 2i$
 (c) $(z - 1), (z - 1 - 2i), (z - 1 + 2i)$
 (d) $f(z) = (z - 1)(z - 1 - 2i)(z - 1 + 2i) = (z - 1)[(z - 1)^2 + 4] = (z - 1)(z^2 - 2z + 5)$
23. (a) $1 + 2i, 1 - 2i, 1, 2$
 (b) $(z - 1 - 2i)(z - 1 + 2i) = (z - 1)^2 - 4i^2 = z^2 - 2z + 5$
 (c) $f(z) = 2(z^2 - 3z + 2)(z^2 - 2z + 5) = 2z^4 - 10z^3 + 26z^2 - 38z + 20$
 (d) Sum = $1 + 2 + 1 + 2i + 1 - 2i = 5$, $S = -\frac{b}{a} = \frac{10}{2} = 5$
 Product = $1 \times 2 \times (1 + 2i) \times (1 - 2i) = 2 \times (1 + 4) = 10$, $P = \frac{e}{a} = \frac{20}{2} = 10$
24. (a) $f(1) = 0 \Rightarrow 2 + a + 26 + b + 20 = 0 \Rightarrow a + b = -48$
 $f(2) = 0 \Rightarrow 32 + 8a + 104 + 2b + 20 = 0 \Rightarrow 8a + 2b = -156 \Rightarrow 4a + b = -78$
 we find $a = -10$ and $b = -38$
 (b) Divide $f(z)$ by $(z - 1)(z - 2) = z^2 - 3z + 2$ and get $2z^2 - 4z + 10 = 2(z^2 - 2z + 5)$
 with roots $1 + 2i, 1 - 2i$
25. The two roots $1 + 2i, 1 - 2i$ give the factor $z^2 - 2z + 5$
 Divide by $z^2 - 2z + 5$ and get $z^2 - 3z + 2$ with roots $1, 2$
26. Factorization finally gives $f(z) = a(z^4 - 5z^3 + 13z^2 - 19z + 10)$
 But $f(-1) = 96 \Leftrightarrow 48a = 96 \Leftrightarrow a = 2$
 Hence $f(z) = 2z^4 - 10z^3 + 26z^2 - 38z + 20$

A. Exam style questions (SHORT)

27. If $(z + 2i)$ is a factor then $(z - 2i)$ is also a factor. $(z + 2i)(z - 2i) = (z^2 + 4)$
 The other factor is $(2z^3 - 3z^2 + 8z - 12) \div (z^2 + 4) = (2z - 3)$
 The other two factors are $(z - 2i)$ and $(2z - 3)$.

28. METHOD 1

If $z = -3 + 2i$ is a root, then $z = -3 - 2i$ is another.

$$\begin{aligned} P(z) &= (z+2)(z-(-3+2i))(z-(-3-2i)) \\ &= (z+2)(z^2 + (3+2i)z + (3-2i)z + 13) \\ &= (z+2)(z^2 + 6z + 13) = z^3 + 8z^2 + 25z + 26 \end{aligned}$$

$$a = 8 \quad b = 25 \quad c = 26$$

METHOD 2

$$0 = -8 + 4a - 2b + c$$

$$0 = 9 + 46i + a(5 - 12i) + b(-3 + 2i) + c$$

$$-9 = 5a - 3b + c \text{ and } 46 = 12a - 2b$$

solving system of three equations $a = 8 \quad b = 25 \quad c = 26$

29.

METHOD 1

Using factor theorem

Substituting $z = -1 - i$ into $P(z)$

$$-(6+n) + (2m-2-n)i = 0$$

Equating both real and imaginary parts to zero

Hence $m = -2$ and $n = -6$

METHOD 2

Using Conjugate root theorem

$$\text{Multiply } (z+1-i)(z+1+i) = z^2 + 2z + 2$$

$$\text{Let } P(z) = (z^2 + 2z + 2)(z-a)$$

$$-2a = -8 \quad a = 4$$

Hence $m = -2$ and $n = -6$

B. Exam style questions (LONG)

30. (a) $(1+i)^2 = 1 + 2i + i^2 = 2i$

(b) $(1+i)^{4n}$

Let $P(n)$ be the proposition: $(1+i)^{4n} = (-4)^n$

We must first show that $P(1)$ is true.

$$(1+i)^4 = ((1+i)^2)^2 = (2i)^2 = 4i^2 = (-4)^1$$

Next, assume that for some $k \in \mathbb{N}^+$

$P(k)$ is true, then show that $P(k+1)$ is true.

$$P(k): (1+i)^{4k} = (-4)^k$$

$$\text{Now, } (1+i)^{4(k+1)} = (1+i)^{4k} (1+i)^4$$

$$= (-4)^k (-4) = (-4)^{k+1}$$

Therefore, by mathematical induction $P(n)$ is true for all $n \in \mathbb{N}^+$

(c) $(1+i)^{32} = (1+i)^{4(8)} = (-4)^8 = 65536$

31. theoretical

32. theoretical