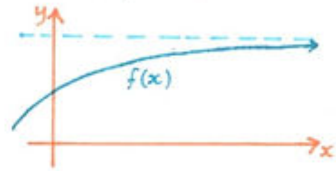
TOPIC 5 CALCULUS

	SL	HL
A&A	✓	✓
A&I	✓	✓

5.1 - Limits

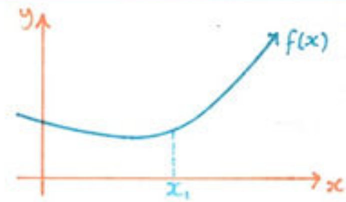
→ We have seen *limits* a couple of times in this course, in functions; we saw graphs that approached an *asymptote*:



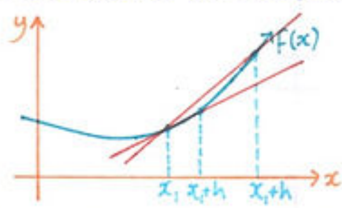
This is the *limit* of $f(x)$. We also saw the *limit* of the sum of a geometric sequence, if $|r| < 1$.

In both of these cases, we never actually reach the limit, we just get infinitely close, and never go beyond it. Limits will be v. important in calculus.

SLOPES OF CURVES → What is the slope of this curve? Well, it *varies*. We could maybe ask what the slope is at x_1, but that is still quite complicated.



Previously, we needed *two points* if we wanted to calculate the slope. So maybe we could choose another point $[x_1+h]$ and find a slope from there. It still doesn't seem very accurate, and it looks like if we move x_1+h closer



to x, this straight lines becomes more appropriate. In fact, if we just consider the *limit* of this slope as *h tends to 0*, that line has the correct slope.

i.e.: $\left[\text{The slope/gradient function}\right] = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{(x+h)-x}\right) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

→ Now, actually evaluating limits requires some more advanced maths, but we will actually just learn rules for various types of functions *(in SL)*.

→ As $h \to 0$, the straight line that is used to find the slope is called a *tangent*: a line that only touches the curve once, at x_1. When that formula is evaluated, the result is a *function for the gradient of the tangent* which will vary depending on the x-value considered.

NOTATION → Given a function, this gradient function is called: *the derivative*

FUNCTION	$f(x) = \sim\sim\sim$	$y = \sim\sim\sim$
DERIVATIVE	$f'(x) = \sim\sim\sim$	$dy/dx = \sim\sim\sim$

You can see the $\frac{\text{rise}}{\text{run}}$ connection here.

5.2 - Increasing/Decreasing functions

	SL	HL
A&A	✓	✓
A&I	✓	✓

DIRECTION OF SLOPE ↦ As you may remember from studying linear functions, any *increasing function* [↗] has a *positive slope*, and any *decreasing function* [↘] has a *negative slope*, and a *flat/horizontal line* [⟶] has *slope = 0*. When we had linear functions, those were *constant slopes*, but the same theory applies to *gradient functions*, and due to this, we can have curves that switch between inc. and dec. : [⌒↘]

> When $f'(x)>0$, graph is increasing
> When $f'(x)<0$, graph is decreasing
> When $f'(x)=0$, graph is flat

↦ We can use this to assess whether functions are increasing/decreasing, or which *sections* of the graph are inc./dec..

↦ Finding when $f'(x)=0$ can tell you when the graph is flat, which actually can tell you where a *maximum* or *minimum* is located. [more in 5.8]

E.G.1 ↦ Is $f(x)=e^{2x}$ an increasing or decreasing function, or both?

↳ The derivative of $f(x)=e^{2x}$ is $f'(x)=2e^{2x}$, finding this will be covered in 5.6. So we can check whether $2e^{2x}$ is positive, negative, or both. Whatever x we choose (-ve, 0, +ve, etc), it always gives a positive value. This means it is an <u>increasing function</u>, which we see in its graph:

E.G.2 ↦ At what interval is $f(x)=x^2-4x+2$ increasing?

↳ Finding this derivative will be covered in 5.3, but I can tell you it is $f'(x)=2x-4$. So we want to know when $2x-4>0 \rightarrow 2x>4 \rightarrow x>2$.
So, to the right of $x=2$, i.e. when $\underline{x>2}$, $f(x)$ is increasing.

E.G.3 ↦ When is $g(x)=\cos x$ decreasing, for $0 \leq x \leq 2\pi$?

↳ Again, the derivative of this is covered in 5.6, but $g'(x)=-\sin x$, so we solve $-\sin x<0 \rightarrow \sin x>0$. As $\sin x$ is the y-coordinate in the unit circle, we know that it is +ve between 0 & π. So $g(x)$ is decreasing at $\underline{0<x<\pi}$.

5.3 - Polynomial derivatives

	SL	HL
A&A	✓	✓
A&I	✓	✓

↦ As we said, you won't need to use the formula using limits, but we will use it here to derive a rule for *derivatives of polynomials*.

E.G.1 ↦ Find the derivative of $f(x) = x^2 - 4x + 2$:

↳ So we will use the 'first principles' formula this one time:

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{(x+h)^2-4(x+h)+2-(x^2-4x+2)}{h}\right) = \lim_{h\to 0}\left(\frac{x^2+2xh+h^2-4x-4h+2-x^2+4x-2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2xh+h^2-4h}{h}\right) = \lim_{h\to 0}(2x-4+h) = \underline{\underline{2x-4}} \text{ as } h\to 0$$

↦ This same process can be applied to a general $ax^n + bx^m + \dots$, and what we get is:

$$\boxed{f(x) = ax^n + bx^m + \dots \Rightarrow f'(x) = anx^{n-1} + bmx^{m-1} + \dots}$$

↦ That can be described as taking each term and *multiplying it by the exponent, then subtracting one from the exponent*. Indeed, in E.G.1, we did go from $x^2 - 4x + 2$ to $(2)x^{2-1} - 4(1)x^{1-1} + 2(0)x^{0-1} = 2x - 4$

FORMULA BOOKLET ↦ In the formula booklet, we are only given this formula:

$$\boxed{f(x) = x^n \Rightarrow f'(x) = nx^{n-1}} \leftarrow \text{In F.B.}$$

So you need to know that the constant does not affect this rule, and that we can seemlessly add terms using this rule for each term.

DIFFERENTIATION ↦ This is simply another term used for the process of finding the derivative.

E.G.1 ↦ Differentiate $f(x) = 4x^7$:

↳ If $f(x) = 4x^7$, then $f'(x) = 4(7)x^{7-1} = 28x^6$

E.G.2 ↦ Find the derivative of $y = 3x^3 - 12x^2 + 8x + 1$:

↳ We use the 'anx^{n-1}' rule again. So $\frac{dy}{dx} = 3(3)x^{3-1} - 12(2)x^{2-1} + 8(1)x^{1-1} + 0$, so $\frac{dy}{dx} = 9x^2 - 24x + 8$

E.G.3 ↦ Find the gradient of the tangent of $f(x) = 5x^4 + x^2$ at $x = 2$:

↳ Find the derivative (the gradient function), then plug in $x = 2$.

$f'(x) = 20x^3 + 2x \longrightarrow f'(2) = 20(2)^3 + 2(2) = 160 + 4 = 164$

5.4 – Tangent/Normal

	SL	HL
A&A	✓	✓
A&I	✓	✓

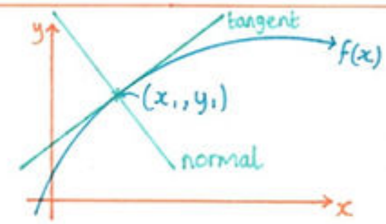
↦ We have seen that the *tangent* is what is actually being assessed when we are finding the *slope at a point*. Well, instead of just finding the slope of it, a natural extension would be to ask what the full *equation of the tangent* is.

TANGENT PROCESS ↦ From 2.1, we know if we have an (x_1, y_1) point on the line, and its slope, we can use $y - y_1 = m(x - x_1)$ to get the equation. So, if we are asked to find the equation of the tangent, given $f(x)$ & an x_1, we can:
① Do $f(x_1)$ to find y_1 // ② Find $f'(x)$ by differentiating // ③ Do $f'(x_1)$ to find m //
④ Plug x_1, y_1 & m into $y - y_1 = m(x - x_1)$ and rearrange if needed

---

E.G. 1 ↦ Find the equation of the tangent of $f(x) = 2x^2 - 6x + 8$ at $x = 2$:
↳ ① $f(2) = 2(2)^2 - 6(2) + 8 = 4$ ∴ we have $(x_1, y_1) = (2, 4)$
② $f'(x) = 4x - 6$ ③ $f'(2) = 4(2) - 6 = 2$ ∴ $m = 2$
④ $y - y_1 = m(x - x_1) \Rightarrow y - 4 = 2(x - 2) \Rightarrow y - 4 = 2x - 4 \Rightarrow \underline{y = 2x}$

NORMAL ↦ The *normal* to a curve is a straight line, passing through a given point, but *perpendicular* to the tangent (see right)



PROCESS ↦ We can still simply find an (x_1, y_1) and the appropriate gradient, we just use the *negative reciprocal* $\left[\frac{-1}{m}\right]$ instead, as it is perpendicular.

E.G. 2 ↦ With $f(x)$ and point from E.G. 1, find the equation of the normal:
↳ We still have $f(2) = 4$ & $(x_1, y_1) = (2, 4)$. $f'(2) = 2$ still, but we use $\frac{-1}{m} = \frac{-1}{2}$ instead. $y - 4 = -\frac{1}{2}(x - 2) \Rightarrow y - 4 = -\frac{1}{2}x + 1 \Rightarrow \underline{y = -\frac{1}{2}x + 5}$

E.G. 3 // IB 2008 ↦ Let $f(x) = e^x(1 - x^2)$. *Part a) showed that* $f'(x) = e^x(1 - 2x - x^2)$,
... d) L is the normal at $P(0, 1)$, show L has equation: $x + y = 1$:
↳ We have $(x_1, y_1) = (0, 1)$, so we need m: for this, we do $f'(x_1) = f(0) = e^0(1 - 2(0) - (0)^2) = 1(1) = 1$. So we use $\frac{-1}{m} = \frac{-1}{1} = -1$. Then, use $y - y_1 = \frac{-1}{m}(x - x_1) \Rightarrow y - 1 = -1(x - 0) \Rightarrow y - 1 = -x \Rightarrow \underline{x + y = 1}$

5.5 - Intro to Integration

	SL	HL
A&A	✓	✓
A&I	✓	✓

⊩ For the majority of operations & processes in Maths, we like to know how to do it in *reverse*. So here, we learn about the opposite of differentiation, known as *integration/anti-differentiation*.

$$\begin{tikzcd} \boxed{\text{FUNCTION}} \arrow[r, bend left, "\text{Differentiation}"] & \boxed{\text{DERIVATIVE}} \arrow[l, bend left, "\text{Integration or Anti-diff.}^{\text{n}}"] \end{tikzcd}$$

⊩ So, again, we will learn rules for integrating functions, starting with polynomials. This can be viewed as going from a function to its *integral* OR from a derivative back to its original function.

NOTATION ⊩ The notation for *the integral of $f(x)$* uses $\int f(x)\,dx$. The 'dx' part of it simply means *with respect to x*.

RULE ⊩ To integrate functions in the form $ax^n + bx^m + \ldots$, we simply do the opposite of multiplying by the power, then subtracting 1 from the power, which is: Adding 1 to the power, then dividing by that new power. i.e.: $\boxed{\int (ax^n)\,dx = \frac{ax^{n+1}}{n+1}}$ → In F.B.

E.G. 1 ⊩ If $f'(x) = 6x^2 + x + 7$, find $f(x)$:

↳ $\int 6x^2 + x + 7\,dx = \frac{6x^{2+1}}{2+1} + \frac{x^{1+1}}{1+1} + \frac{7x^{0+1}}{0+1} + C = \underline{\underline{2x^3 + \frac{x^2}{2} + 7x + C}}$ explained below

E.G. 2 ⊩ Find $\int 10x - 6\sqrt{x}\,dx$:

↳ $\int 10x^1 - 6x^{1/2}\,dx = \frac{10x^{1+1}}{2} - \frac{6x^{1/2+1}}{3/2} + C = \underline{\underline{5x^2 - 4x^{3/2} + C}}$

BOUNDARY CONDITION ⊩ We will now explain the '+c' at the end of the integral. If you consider that any constant disappears when differentiated, then when we are integrating, we must write '+c' at the end, in case there was any constant in the original function.

FINDING c ⊩ If we want to find this *mystery constant*, then we must know a set of coordinates that fit the original function. We can then plug them in and *solve for c*. We can then write out the whole function

$\boxed{5.5}$ continued

FINDING C ↦ Here are some examples of that process:

E.G.3 ↦ If $\frac{dy}{dx} = 3x^2 + x$ and $y = 10$ when $x = 1$, find y:

↳ Integrate: $y = x^3 + \frac{x^2}{2} + c$, plug in $(1,10)$: $10 = (1)^3 + \frac{(1)^2}{2} + c \Rightarrow c = 8.5$

Write full equation: $\underline{y = x^3 + \frac{x^2}{2} + 8.5}$

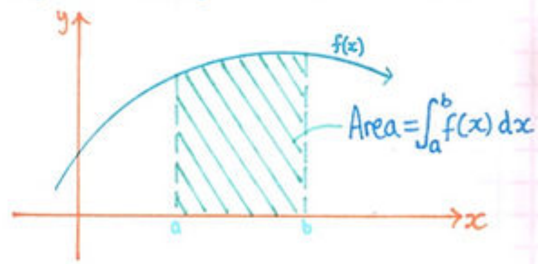
E.G.4 ↦ If $\int f(z)\,dz = F(z)$, $f(z) = 15z^4 - 10z^{-2}$ and $F(2) = 73$, find $F(z)$:

↳ Integrate $\int 15z^4 - 10z^{-2}\,dz = 3z^5 + 10z^{-1} + c$. Plug in $(2, 73) \Rightarrow$

$3(2)^5 + \frac{10}{(2)} + c = 73 \Rightarrow 96 + 5 + c = 73 \Rightarrow c = -18 \therefore \underline{F(z) = 3z^5 + \frac{10}{z} - 18}$

↦ Note - they won't tell you when you will be expected to find c as well.

DEFINITE INTEGRALS ↦ When we do $\int f(x)dx$, we get a function as a result, and this is called an *indefinite integral*. A *definite integral* is, instead, in the form: $\int_a^b f(x)\,dx$. Here, we get a number as a result of evaluating the integral in the *interval* $a \leq x \leq b$. This has one main use, which is that



it gives us the area between $f(x)$ and the x-axis, between a & b. See above ↑

The theory behind this involves splitting up this area into thin rectangles, and evaluating the *limit* as the width $\rightarrow 0$. But this theory is beyond the scope of the SL course.

↦ You will learn how to do this manually will be covered in $\boxed{5.11}$, but if it is a complicated function, in Paper 2, your GDC can find the definite integral, and therefore, the *area under the curve*.

↦ In $\boxed{5.11}$, we will also cover what happens if some of the curve is below the x-axis, and how to find the area between two curves.

TI-nspire ↦ Open 'CALCULATOR' → [⇧SHIFT] [+] → Enter limits and function

TI-84 ↦ [MATH] → [9: fnInt] → Enter limits and function

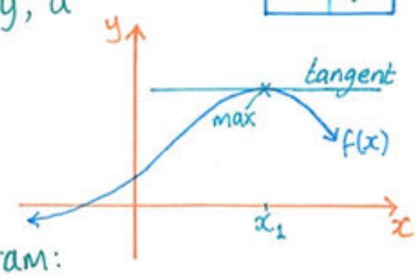
E.G.5 ↦ Find the area enclosed by $f(x) = e^{\cos(2x)}$, the x-axis, y-axis and $x = 3$:

↳ Need to evaluate $\int_0^3 e^{\cos 2x}\,dx$. Use GDC, area = $\underline{3.60}$ units²

5.6 – Turning Points

	SL	HL
A&A	(5.8)	(5.8)
A&I	✓	✓

TURNING POINTS // As has been mentioned already, a maximum or minimum point on a curve is the point which has a flat/horizontal tangent. In general, a flat line has slope = 0. Therefore, if $f'(x_1)=0$, then x_1 is a turning point, as in the diagram:



E.G.1 // Find the turning point of $f(x)=3x^2-12x+4$:

↳ Differentiate: $f'(x)=6x-12$. Set $f'(x)=0$: $6x-12=0$

$6x=12 \rightarrow x=2$. Find y: $f(2)=3(2)^2-12(2)+4=-8$. $\therefore$ $\underline{(2,-8)}$

E.G.2 // Find the stationary points of $y=x^3-3x^2+4$:

↳ $\frac{dy}{dx}=3x^2-6x=0 \rightarrow 3x(x-2)=0 \rightarrow x=0$ & 2. Find y's: $y=(0)^3-3(0)^2+4=4$,

$y=(2)^3-3(2)^2+4=0$ $\therefore$ Stationary points at $\underline{(0,4)}$ & $\underline{(2,0)}$.

TESTING FOR MAX./MIN. // In the other courses, there are ways of finding out whether it is indeed a maximum or a minimum point. This could be the 2nd derivative test, but we don't need that in A&I SL. Instead, you could work it out by graphing on your calculator, or your knowledge of functions. e.g.: ∪ or ∩-shaped quadratics.

E.G.3 // Find & classify turning point of $f(x)=-3x^2-12x+8$ using calculus:

↳ Firstly, we find the vertex by setting $f'(x)=0$. $f'(x)=-6x-12$.

$-6x-12=0 \rightarrow -6x=12 \rightarrow x=-2$. $f(-2)=-3(-2)^2-12(-2)+8=20$. So $\underline{(-2,20)}$ is the vertex. As $a<0$, it is an ∩-shaped quad., so a $\underline{\text{maximum point}}$.

E.G.4 // IB 2019 // $f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2+kx+5$. Local maximum at $x=-3$.

a) Show $k=-6 \rightarrow x^2+x+k=f'(x)$. $f'(-3)=0$ because it is a max. point. $(-3)^2-3+k=0 \rightarrow 9-3+k=0 \rightarrow k=-6$.

b) Find the coordinates of the minimum: $\rightarrow x^2+x-6=0 \rightarrow (x+3)(x-2)=0$

$\rightarrow x=2$ or $\cancel{-3}$. $f(2)=\frac{1}{3}(2)^3+\frac{1}{2}(2)^2-6(2)+5=\underline{-\frac{7}{3}}$

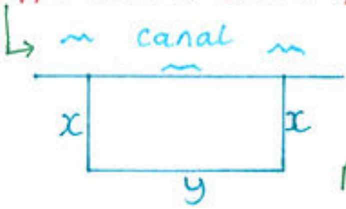
5.7 - Optimisation

	SL	HL
A&A	(5.8)	(5.8)
A&I	✓	✓

OPTIMISATION // → Now that we know how to find the maximum and minimum points of some functions, then we can do the same for functions that are specifically related to real-world situations. Finding the turning points of these real-world functions, is called optimisation. This can involve certain problems, such as: minimising costs, maximising an area/volume, maximising profits, etc. One of the trickiest parts of these questions, is creating the function in the first place, if it is not given.

E.G. 1 // → A box has sides: $x, 2x, 18-3x$. Find the x that minimises the volume: → First, we make a function for volume (V): $V = x \cdot 2x \cdot (18-3x) = 36x^2 - 6x^3$. Find $\frac{dV}{dx}$: $\frac{dV}{dx} = 72x - 18x^2$. Solve, $=0$: $72x - 18x^2 = 0$ → $18x(4-x) = 0$ ∴ $18x = 0$ or $(4-x) = 0$ → $x = 0$ or 4. But you can't have side length of 0, so $\underline{x = 4}$.

E.G. 2 // → 3 sides of a rectangular area next to a canal are being fenced off. There is 120m of fence. What side lengths would make the largest area?



Area $= xy$ // Perimeter: $120 = 2x + y \rightarrow y = 120 - 2x$

Substitute: $A = x(y) \rightarrow A = x(120-2x) \rightarrow A = 120x - 2x^2$

Maximise: $\frac{dA}{dx} = 120 - 4x = 0 \rightarrow \underline{x = 30m}$, ∴ $\underline{y = 60m}$

E.G. 3 // IB 2016 // → Selling juice in cylindrical cans, each with volume of $340 cm^3$. Surface area is given by $A = 2\pi r^2 + \frac{680}{r}$, where r is the radius, in cm. To reduce cost, we minimise area.

a) Find $\frac{dA}{dr}$ → $A = 2\pi r^2 + \frac{680}{r}$ can be rewritten: $A = 2\pi r^2 + 680r^{-1}$

$\frac{dA}{dr} = 2(2\pi r^1) + (-1)680r^{-2} = \underline{4\pi r - 680r^{-2}}$ or $4\pi r - \frac{680}{r^2}$

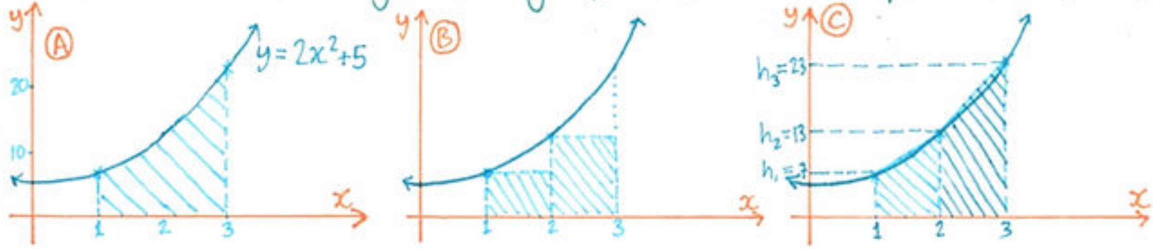
b) Find r that minimises S.A. → $4\pi r - \frac{680}{r^2} = 0$ multiply by r^2:

$4\pi r^3 - 680 = 0 \rightarrow 4\pi r^3 = 680 \rightarrow \pi r^3 = 170 \rightarrow r^3 = 54.1127 \rightarrow \underline{r = 3.78 cm}$

5.8 – Trapezoidal Rule

	SL	HL
A&A		
A&I	✓	✓

TRAPEZOIDAL RULE// → So you already know, that you can find the exact area under a curve, using definite integrals. i.e.: area between $f(x)$ & x-axis between $x=a$ & $x=b$, is $\int_a^b f(x)\,dx$. However, sometimes you may not be given the equation, or you won't have the tools to integrate it. So how could we *approximate* it instead? Well, integrals are already technically using infinitely thin rectangles (in Ⓐ). Ⓑ estimates area using 2 rectangles, but Ⓒ uses 2 trapezoids (*trapezia*).



// → If we now compare the approximations in Ⓑ & Ⓒ to the actual area found in Ⓐ, we see the benefit of *trapezoids (trapezia)*:

Ⓐ: $\int_1^3 (2x^2+5)\,dx = \left[\frac{2x^3}{3}+5x\right]_1^3 = (18+15)-(\tfrac{2}{3}+5) = \underline{27\tfrac{1}{3}}$

Ⓑ: Rectangles $= (\omega \times h_1) + (\omega \times h_2) = (1\times 7) + (1 \times 13) = \underline{20}$

much closer!

Ⓒ: Trapezoids $= (\tfrac{1}{2}\omega(h_1+h_2)) + (\tfrac{1}{2}\omega(h_2+h_3)) = \tfrac{1}{2}(7+13) + \tfrac{1}{2}(13+23) = \underline{28}$

RULE// → So, we use this *trapezoidal rule*. The area of a trap. is usually given by $A=\tfrac{1}{2}(b_1+b_2)h$, but maybe '$A=\tfrac{1}{2}(h_1+h_2)\omega$' makes more sense here. Sometimes you will be given the function or graph, sometimes just a table of x & y values:

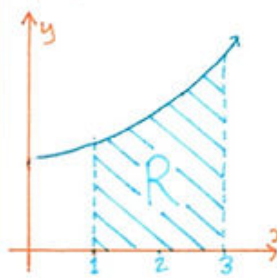
E.G.1// → Estimate the area under $f(x)$, between 0 & 3, using 5 trapezoids:

x	$f(x)$	AREA:
0	8	
		$A_1=\tfrac{1}{2}(8+7.24)(0.6)=4.572$
0.6	7.24	
		$A_2=\tfrac{1}{2}(7.24+6.86)(0.6)=4.23$
1.2	6.86	
		$A_3=\tfrac{1}{2}(6.86+6.52)(0.6)=4.014$
1.8	6.52	
		$A_4=\tfrac{1}{2}(6.52+6.6)(0.6)=3.936$
2.4	6.6	
		$A_5=\tfrac{1}{2}(6.6+7.6)(0.6)=4.26$
3	7.6	

TOTAL AREA = 21.0 units²

E.G.2// → a) Complete the table for $f(x)=\sqrt{2^x+1}$:

x	1	1.5	2	2.5	3
$f(x)$	1.732	1.957	2.236	2.693	3



b) Hence, use trapezoidal rule to estimate area R:

↳ $\tfrac{1}{2}\cdot\tfrac{1}{2}(1.732+2\cdot 1.957+\ldots$
$\ldots+2\cdot 2.836+2\cdot 2.693+3)$

$=\tfrac{1}{4}(18.504) \approx \underline{4.63}$ units²

c) Is this an over- or under-estimate?

↳ Traps. have height <u>above</u> curve. So: over-estimate

5.9 – Differentiation Rules [AHL]

	SL	HL
A&A		5.6
A&I		✓

↦ In A&I SL calculus, we only learned how to differentiate functions with terms in the form $a_n x^n$, such as *polynomials*. In this section, we learn how to differentiate lots of other functions:

RULES ↦ With enough time, you could derive these using *first principles*, but in practice you will just use this list of specific derivatives as fact, and use them moving forward to solve problems.

$f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$	$e^x \Rightarrow e^x$
$\sin x \Rightarrow \cos x$	$\ln x \Rightarrow \frac{1}{x}$
$\cos x \Rightarrow -\sin x$	$\tan x \Rightarrow \frac{1}{\cos^2 x}$

→ In F.B.

NOTE ↦ Remember about *sum of terms* & *scalar multiples* of terms: $\boxed{(f+g)'=f'+g' \quad (af)'=af'}$

NOTATION ↦ If you have $\boxed{y=f(x) \Rightarrow \frac{dy}{dx}=f'(x)}$

E.G.1 ↦ Find the derivative of $f(x)=6x^{4/3}$:

↳ $f'(x)=\frac{4}{3}\cdot 6x^{1/3} = \underline{8(\sqrt[3]{x})}$

E.G.3 ↦ Differentiate $g(x)=7\sin x-\ln x$:

↳ $g'(x)=\underline{7\cos x-\frac{1}{x}}$

E.G.2 ↦ Differentiate $y=5e^x$:

↳ $\frac{dy}{dx}=\underline{5e^x}$

E.G.4 ↦ If $y=2-\frac{1}{2}\tan x$, find $\frac{dy}{dx}$:

↳ $\frac{dy}{dx}=-\frac{1}{2}\left(\frac{1}{\cos^2 x}\right)=\underline{\frac{-1}{2\cos^2 x}}$

PRODUCT RULE ↦ So far, we have only seen rules involving sums of terms or scalar multiples of terms, but what about when you are given *a product of two functions*?

$$\boxed{\frac{d}{dx}\left(f(x)\cdot g(x)\right)=f(x)g'(x)+g(x)f'(x)} \text{ OR } \boxed{\frac{d}{dx}(u\cdot v)=u\cdot\frac{dv}{dx}+v\cdot\frac{du}{dx}}$$

→ In F.B.

↦ It is a good habit to list out $u, u', v,$ & v' before carrying out the *product rule*.

E.G.5 ↦ Differentiate $y=(x^2+3x)(2x-1)$:

$u=x^2+3x$	$v=2x-1$
$u'=2x+3$	$v'=2$

$\therefore \frac{dy}{dx}=(x^2+3x)(2)+(2x-1)(2x+3)=\underline{6x^2+10x-3}$

E.G.6 ↦ $h(x)=x^2\sin x$, find $h'(x)$:

$f(x)=x^2$	$g(x)=\sin x$
$f'(x)=2x$	$g'(x)=\cos x$

$\therefore h'(x)=\underline{x^2\cos x+2x\sin x}$

E.G.7 ↦ Find the gradient of the tangent of $y=x\ln x$ at $x=1$:

$u=x$	$v=\ln x$
$u'=1$	$v'=1/x$

$\therefore \frac{dy}{dx}=x\left(\frac{1}{x}\right)+(\ln x)\cdot(1)=\frac{x}{x}+\ln x=1+\ln x$ // At $x=1$, slope $=1+\ln 1=1+0=\underline{1}$

QUOTIENT RULE ↦ So what about for one function *divided* by another?

Be careful with the order for this rule

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}} \text{ OR } \boxed{\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}}$$

→ In F.B.

E.G.8 ↦ If $y=\frac{3x-2}{2x-5}$, find $\frac{dy}{dx}$:

$u=3x-2$	$v=2x-5$
$u'=3$	$v'=2$

$\therefore \frac{dy}{dx}=\frac{(2x-5)(3)-(3x-2)(2)}{(2x-5)^2}=\frac{6x-15-6x+4}{(2x-5)^2}=\underline{\frac{-11}{(2x-5)^2}}$

5.9 continued

CHAIN RULE ↦ The third of our big rules is for *composite functions* – a function *within* another function

$$\boxed{y = g(u), \text{ where } u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$ → In F.B.

↦ But this is a fairly confusing way of presenting it, so may find this more manageable:

$$\boxed{\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)}$$ → Put $g(x)$ into the derivative of $f(x)$, then multiply by the derivative of $g(x)$.

E.G.9 ↦ Take $y = \cos(3x)$, find $\frac{dy}{dx}$:

↳ $\frac{dy}{dx} = -\sin(3x) \cdot 3 = \underline{-3\sin(3x)}$

E.G.10 ↦ $f(x) = \sqrt{3x^2+5x}$, find $f'(x)$:

↳ $f(x) = (3x^2+5x)^{1/2}$, $f'(x) = \frac{1}{2}(3x^2+5x)^{-1/2} \cdot (6x+5) = \underline{\frac{6x+5}{2\sqrt{3x^2+5x}}}$

E.G.11 ↦ Differentiate $y = \sin^2 x$, and simplify:

↳ $y = (\sin x)^2$, $\frac{dy}{dx} = 2(\sin x) \cdot \cos x = 2\sin x \cos x = \underline{\sin 2x}$

E.G.12 ↦ If $g(x) = 3\ln(5x+2)$, find $g'(x)$:

↳ $g'(x) = 3 \cdot \left(\frac{1}{5x+2}\right) \cdot 5 = \underline{\frac{15}{5x+2}}$

RELATED RATES ↦ We might then see some situations where three variables are all changing at various rates, in relation to the other variables. This is essentially just another use of the *chain rule* $\left[\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}\right]$, almost like how, in general, $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$. This requires info about relationships between the 3 variables, maybe using some geometry formulae. Also consider that: If $\frac{dA}{dB} = \frac{a}{b}$, then $\frac{dB}{dA} = \frac{b}{a} = \frac{1}{b/a}$.

E.G.13 ↦ A circle is growing at a rate of 20cm/s. Find the rate of change of the area at the point when radius = 50cm?

↳ The rate of change of the area is given by $\frac{dA}{dt}$. However, we only know A in terms of r: $A = \pi r^2$. So we use $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$, and differentiate A to find: $\frac{dA}{dr} = 2\pi r$. We were also told that $\frac{dr}{dt} = 20 \therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \times 20 = 40\pi r$. So when $r = 50$, $\frac{dA}{dt} = 40\pi(50) = 2000\pi \approx \underline{6280 \text{ cm}^2/\text{s}}$

E.G.14 ↦ Air is being pumped into a spherical balloon at a rate of $5 \text{cm}^3/\text{min}$. At what rate is the radius increasing when the diameter is 20cm?

↳ The rate of change of the radius is given by $\frac{dr}{dt}$. If $d = 20$, then $r = 10$. We know V in terms of r, which is $V = \frac{4}{3}\pi r^3$. So we can find $\frac{dV}{dr} = 4\pi r^2$. We are told $\frac{dV}{dt} = 5$. Then $\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$ by the chain rule. So if $\frac{dV}{dr} = 4\pi r^2$, then $\frac{dr}{dV} = \frac{1}{4\pi r^2}$. So $\frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot 5 = \frac{5}{4\pi r^2}$. When $r = 10$, $\frac{dr}{dt} = \frac{5}{400\pi} = \underline{\frac{1}{80\pi} \text{ cm/min}}$

E.G.15 ↦ Gas is escaping from a spherical ball at $2\text{ft}^3/\text{min}$. How fast is the surface area shrinking when the radius of the balloon is 12ft?

↳ We have two separate equations: ① $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$, ② $\frac{dSA}{dt} = \frac{dSA}{dr} \cdot \frac{dr}{dt}$. For a sphere, we also know that $V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$. Therefore, $\frac{dV}{dr} = 4\pi r^2$ and $\frac{dSA}{dr} = 8\pi r$. We are also told $\frac{dV}{dt} = 2$. Therefore, if $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \rightarrow 2 = 4\pi r^2 \cdot \frac{dr}{dt}$ → so $\frac{dr}{dt} = \frac{1}{2\pi r^2} = \frac{1}{288\pi}$. Plug this into equation ②: $\frac{dSA}{dt} = 8\pi r \cdot \frac{1}{288\pi} = \frac{96\pi}{288\pi} = \underline{\frac{1}{3} \text{ ft}^2/\text{min}}$

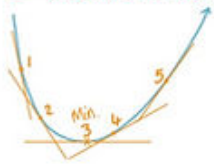
5.10 – Second Derivatives [AHL]

	SL	HL
A&A	5.7/8	5.7/8
A&I		✓

MEANING → If you differentiate a function once, you have the *first derivative*. If you then differentiate the first derivative again, then you have the *second derivative*. All the techniques learned before still apply.

RESULT → If the first derivative gives you a *slope function*, or an *instanteous rate of change function*, then the second derivative tells you the *rate of change of the derivative*. So as you move to the right on a graph, the 2nd deriv. will tell you whether the slope is increasing or decreasing, and *how quickly*. NOTATION → $y=f(x) \Rightarrow \frac{dy}{dx}=f'(x) \Rightarrow \frac{d^2y}{dx^2}=f''(x)$

2nd DERIVATIVE TEST → The diagram on the right shows 5 points on a curve, around a minimum point, with their tangents.



We see that the slopes go from negative to zero to positive, which is increasing. And increasing slopes also means the second derivative is positive on this curve. This would be the case around any minimum point. And the opposite is true for any maximum point:

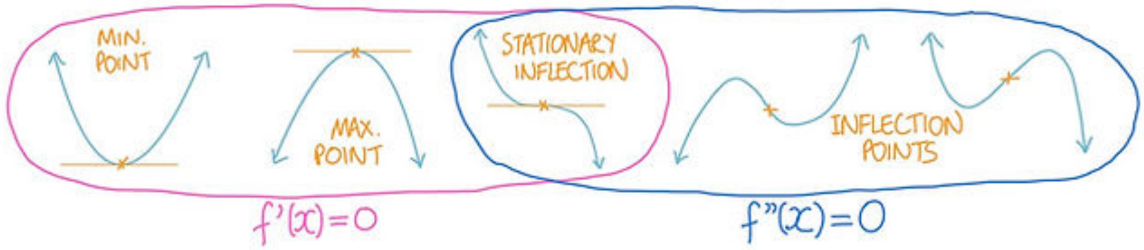
If x_1 is a min. point $\Rightarrow f'(x_1)=0 \Rightarrow f''(x_1)>0 \Rightarrow$ Curve is "concave up"

If x_2 is a max point $\Rightarrow f'(x_2)=0 \Rightarrow f''(x_2)<0 \Rightarrow$ Curve is "concave down"

E.G.1 → Find and classify extrema for $f(x)=3x^5-25x^3+60x+20$:

↳ $f'(x)=15x^4-75x^2+60=0 \longrightarrow$ solve by GDC (or factoring) $\rightarrow x=\underline{1,-1,2,-2}$. Find $f''(x)$: $f''(x)=60x^3-150x$, then plug in our turning points: $f''(1)=-90<0 \therefore$ MAX, $f''(-1)>0 \therefore$ MIN, $f''(-2)<0 \therefore$ MAX, $f''(2)>0 \therefore$ MIN

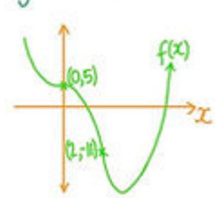
INFLECTION POINTS → That only covers when the 2nd derivative is positive or negative. What about when $f''(x)=0$? This is possible – usually at the point where we switch between concave up & down. This can occur either at a stationary point, or anywhere else on the graph. This is called an *inflection point*, as long as there's *also* a change in concavity.



→ You may need to verify that x_1 is an inflection point, even after finding out that $f''(x_1)=0$. Here, we need to show the change in concavity: plug in x-values either side of x_1 into $f''(x)$ and see if you get 1 +ve and 1 -ve.

E.G.2 → $f(x)=x^4-4x^3+5$ has two inflection points. Find the coordinates of both:

↳ $f'(x)=4x^3-12x^2$. Then $f''(x)=12x^2-24x$. Set $f''(x)=0 \rightarrow 12x^2-24x=0 \rightarrow 12x(x-2)=0$
$\rightarrow x=0$ & 2. Do $f(0)$ & $f(2)$ to find y-coordinates, so we get $\underline{(0,5)}$ & $\underline{(2,-11)}$



E.G.3 → Verify that (2,3) is an inflection point of $y=x^3-6x^2+12x-5$:

↳ ① Check $\frac{d^2y}{dx^2}=0$: $\frac{dy}{dx}=3x^2-12x+12 \rightarrow \frac{d^2y}{dx^2}=6x-12$. Plug in $x=2$: $6(2)-12=0$ ✓

② Change in concavity: Plug in $x=1$ into 2nd deriv. $\rightarrow 6(1)-12<0 \therefore$ concave down

Plug in $x=3 \rightarrow 6(3)-12>0 \therefore$ Concave up. So change in concavity ✓ Inflection point verified.

5.11 – Integration Methods [AHL]

	SL	HL
A&A	5.10	5.10
A&I		✓

BASIC INTEGRALS ↦ To add to the rule seen in 5.5, we now extend it to six types of functions that we'll be able to *integrate*. Along with that, we must remember that $a\int f(x)dx = \int af(x)dx$.

In F.B.

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$	$\int \frac{1}{\cos^2 x} dx = \tan x + C$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\int \sin x\, dx = -\cos x + C$	$\int \cos x\, dx = -\sin x + C$	$\int e^x dx = e^x + C$

E.G.1 ↦ Find $\int 4x^3 + \frac{1}{x}\, dx$:

↳ $\frac{4x^{3+1}}{3+1} + \ln\lvert x\rvert + C = \underline{x^4 + \ln\lvert x\rvert + C}$

E.G.2 ↦ If $\frac{dy}{dx} = 5\sin x$, find y:

↳ $y = \underline{-5\cos x + C}$

"$ax+b$" RULES ↦ Recall what happens when differentiating with the *chain rule* with functions in the form $f(ax+b)$, which is that: $\frac{d}{dx}(\sin(5x+2)) = 5\cos(5x+2)$ or $\frac{d}{dx}((6x-7)^3) = 3(6x-7)^2 \times 6$, i.e.: $\frac{d}{dx}(f(ax+b)) = af'(ax+b)$. The opposite of multiplying by 'a' when differentiating is to *divide by 'a' when integrating $f(ax+b)$*:

$$\boxed{\text{If } \int f(x) = F(x) + C,\ \text{then } \int f(ax+b) = \frac{1}{a} f'(ax+b) + C}$$

NOT in F.B.

E.G.3 ↦ Find $\int 10\cos(5x+3)dx$:

↳ $\left(\frac{1}{5}\right)\cdot 10\sin(5x+3) + C = \underline{2\sin(5x+3) + C}$

E.G.4 ↦ Find $\int_0^1 16e^{5-4x} dx$:

↳ $\left[\left(\frac{1}{-4}\right)\cdot 16e^{5-4x}\right]_0^1 = \left[-4e^{5-4x}\right]_0^1 = (-4e^{5-4(1)}) - (-4e^{5-4(0)}) = \underline{4e(e^4-1)}$

↦ They give you integrals in this form far more often than the basic integrals.

INT. BY SUBSTITUTION ↦ If $\boxed{\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)}$, then also $\boxed{\int f'(g(x))g'(x)dx = f(g(x)) + C}$. This means we should be able to do a *reverse chain rule* for integration, whenever you see a function $g(x)$, and its derivative, $g'(x)$. But what will make it easier is to *substitute* $g(x) = u$, considering $\frac{du}{dx}$, simplifying, then doing $\int f(u)du$:

E.G.5 ↦ Find $\int x^3\sqrt{x^4+1}\, dx$:

↳ Set $u = x^4+1$. Then $\frac{du}{dx} = 4x^3$. Rearrange to $\frac{1}{4}du = x^3 dx$, all this means that $\int \sqrt{x^4+1}\cdot x^3 dx = \int \sqrt{u}\cdot\frac{1}{4}du = \frac{1}{4}\int\sqrt{u}\, du$ $= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C$. Substitute back $u = x^4+1 \longrightarrow \underline{\frac{(x^4+1)^{3/2}}{6} + C}$

E.G.6 ↦ Find $\int \sin^3 x \cos x\, dx$:

↳ Take $u = \sin x$, $\frac{du}{dx} = \cos x$, $du = \cos x\, dx \rightarrow \int u^3 du = \frac{u^4}{4} + C$

Substitute back $\Rightarrow \underline{\frac{\sin^4 x}{4} + C}$

↦ As with any integral problem, you need to spot when they're asking you to find C as well.

E.G.7 ↦ Find the area enclosed by $y = \frac{(\ln x)^2}{x}$, $x=1$, and $x=e$:

↳ We notice this function contains $\frac{1}{x}$ & $\ln x$, so we can use substitution. So take $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$.

Then $\int_1^e \frac{(\ln x)^2}{x} dx = \int_{(1)}^{(e)} u^2 \frac{du}{dx} dx = \int u^2 du = \left[\frac{u^3}{3}\right] \xrightarrow[\text{back}]{\text{sub}} \left[\frac{(\ln x)^3}{3}\right]_1^e = \frac{1^3}{3} - \frac{0^3}{3} = \underline{\frac{1}{3}}$

↦ Here, technically, these limits should be 'u-values' until you substitute back and have everything in terms of x. Although if you do find these 'u' limits, then there wouldn't be any need to substitute back at all.

↦ If you are very comfortable with substitution, then there's actually no need for '$ax+b$ rules'. Just do sub always.

5.12 – Area & Volume [AHL]

	SL	HL
A&A	5.12	5.12/17
A&I		✓

MANUAL PROCESS ⟼ We saw in section 5.5 how to find areas between a curve and the x-axis on a GDC, using *definite integrals*. In this section, we'll recall how to do definite integrals by hand, and therefore how to find areas by hand to. It is also gets extended to areas *below* the x-axis, areas *between curves*, areas between the curve and the *y-axis*, then to *volume* problems too.

$$\boxed{\text{If } \int f(x)dx = F(x)+c, \text{ then } \int_a^b f(x)dx = F(b)-F(a)} \text{ or simply: } \boxed{\int_a^b g'(x)dx = g(b)-g(a)}$$

Meaning: Integrate, plug in b, plug in a, and find the difference:

E.G.1 ⟼ Find the area beneath the curve of $f(x)=\sin x$, between $x=0$ & $\pi/3$:

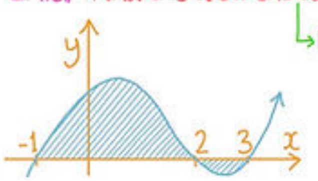
↳ $\int_0^{\pi/3} \sin x \, dx = [-\cos x]_0^{\pi/3} = (-\cos\frac{\pi}{3})-(-\cos 0) = -\frac{1}{2}+1 = \underline{\frac{1}{2}}$

E.G.2 ⟼ Find the area enclosed by $y=x^2+2$, the x-axis, $x=1$, and $x=3$:

↳ $\int_1^3 (x^2+2)dx = \left[\frac{x^3}{3}+2x\right]_1^3 = \left(\frac{27}{3}+2(3)\right)-\left(\frac{1}{3}+2(1)\right) = 15-2\frac{1}{3} = \underline{12\frac{2}{3}}$

CROSSING X-AXIS ⟼ When doing area between the curve and the x-axis, we need to be careful of when there are regions *below the x-axis*. If you remember: integrals are essentially adding up areas of infinitely thin rectangles, using 'y-coordinates' as their 'heights'. So when the curve goes below the x-axis, those become *negative*, even though *area* always stays positive. So we must adjust. If we take the *absolute value* of the regions below the x-axis, or do $\int_a^b |f(x)|dx$ on our GDC, we can ensure the correct area.

E.G.3 ⟼ Find the area enclosed by $f(x)=x^3-4x^2+x+6$ and the x-axis:

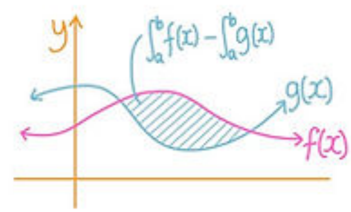


↳ Roots at $x=-1, 2$ & 3 ∴ split the integral and use abs. value. Area $= \left|\int_2^3 f(x)dx\right| + \left|\int_{-1}^2 f(x)dx\right|$

$= \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{x^2}{2}+6x\right]_2^3\right| + \left|\left[\frac{x^4}{4}-\frac{4x^3}{3}+\frac{x^2}{2}+6x\right]_{-1}^2\right| = \left|\frac{45}{4}\right| + \left|\frac{-7}{12}\right| = \frac{45}{4}+\frac{7}{12} = \frac{71}{6} \approx \underline{11.8}$

OR Just type $\int_{-1}^3 |x^3-4x^2+x+6|dx$ on GDC and get $\underline{11.8}$ immediately.

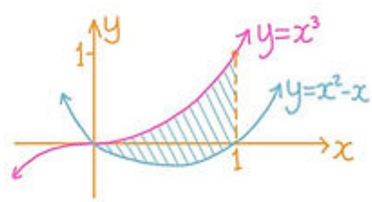
AREA BETWEEN CURVES ⟼ We can find this by using the geometric fact that the area under $f(x)$ minus the area under $g(x)$ leaves us with the *area between* the curves. We can either do 2 separate integrals, or subtract the functions, simplify, and then integrate.



$$\boxed{\text{AREA BETWEEN CURVES} = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x)dx - g(x))dx}$$ ↳ Not in F.B.

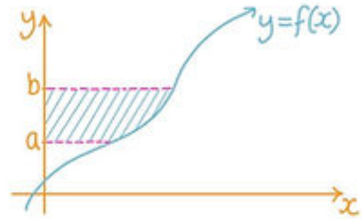
E.G.4 ⟼ Find the area enclosed by $y=x^3$ and $y=x^2-x$, as well as the line $x=1$:

↳ Graphing it on our GDC, we see both that they intersect at $x=0$, and $y=x^3$ is *above*. So Area $= \int_0^1 (x^3-(x^2-x))dx = \frac{5}{12} \approx \underline{0.417}$



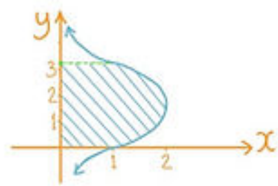
⟼ If you have sections where f & g switch positions, then *absolute value* should again be used.

5.12 continued



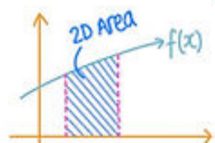
Y-AXIS AREAS → So we've dealt with area *under* a curve, which can also be seen as the area between the curve and the x-axis. But what about the area *between the curve and the y-axis*? As seen here →

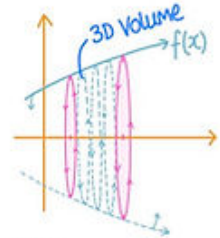
RULE → To do this, we need to regard x as a *function of* y, and then perform $\int_a^b f(y)\,dy$.



E.G.5 → Find the area bounded by $y=\sin^{-1}(x-1)$, $y=0$, $y=\pi$, and the y-axis:
↳ Make x the subject: $\sin y = x-1 \rightarrow x = \sin y + 1$, then integrate this with respect to y: $\int_0^{\pi}(1+\sin y)\,dy = [y-\cos y]_0^{\pi} = (\pi-\cos\pi)-(0-\cos 0) = (\pi-(-1))-(-1) = \underline{\pi+2}$

SOLIDS OF REVOLUTION → Having just dealt with finding 2D areas by considering a sum of infinitely thin rectangles, or trapezia (as in 5.8), then it follows that a similar technique could be used to analyse the *volume of a solid formed by rotating the curve 360° [2π rad.] around the x-axis.*



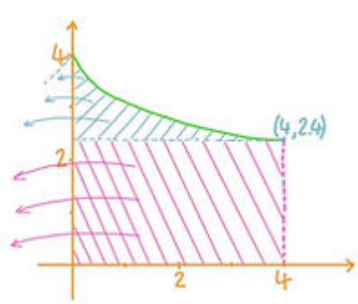


→ Those *rectangles* that we were adding before now become extremely narrow *cylinders* that look like flat discs, being summed together. Briefly: As the volume of a cylinder is given by $V=\pi r^2 h$, and as 'r' is actually the 'y'-coordinate for each cylinder, and as $h \rightarrow 0$, we get the formula:

The volume of the solid created when rotating 2π around the x-axis between a&b: $\boxed{V=\int_a^b \pi y^2\,dx}$ → In F.B.

E.G.6 IB 2015 → The region that is bounded by $y=\ln(5x+10)$, and the lines $x=e$ and $x=2e$ is rotated 2π about the x-axis. Find the volume of this region:
↳ Use: $V=\int_a^b \pi y^2\,dx \Rightarrow \int_e^{2e}\pi\cdot(\ln(5x+10))^2\,dx = \pi\cdot\int_e^{2e}(\ln(5x+10))^2\,dx$. This is definitely too tricky to consider doing this manually. We type this final form into our GDC ... $\approx \underline{99.2}$ (units^3)

→ Again, for this too, you may have to do *volume about the y-axis*. In that case: $\boxed{V=\int_a^b \pi x^2\,dy}$ → In F.B.



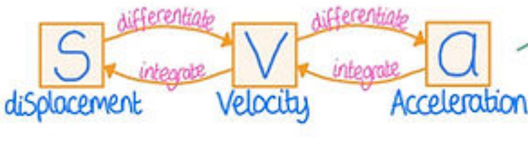
E.G.7 → The volume of a candle can be modelled by rotating the area under the curve $y=2\left(\frac{x+2}{x+1}\right)$, for the domain of $0 \leq x \leq 4$, 360° about the y-axis. Find the volume, where this measured in cms:
↳ Firstly, we'll need to make x the subject: $\frac{y}{2}=\frac{x+2}{x+1} \rightarrow \frac{y}{2}(x+1)=x+2$
$\rightarrow \frac{xy}{2}+\frac{y}{2}-2=x \rightarrow \frac{y}{2}-2=x-\frac{xy}{2} \rightarrow \frac{y}{2}-2=x\left(1-\frac{y}{2}\right) \rightarrow \frac{\frac{1}{2}y-2}{1-\frac{1}{2}y}=x$
$\rightarrow x=\frac{4-y}{y-2}$. Our limits need to be y-values: at $x=0, y=4$, and at $x=4, y=2.4$. But if we do $\int_{2.4}^{4}\pi x^2\,dy$, we'd just get the blue volume. We can also rotate the line $x=4$ about the y-axis to get the pink volume. So on our GDC, we do two integrals:
$\text{Vol.}=\int_{2.4}^{4}\pi\left(\frac{4-y}{y-2}\right)^2 dy + \int_0^{2.4}\pi(4)^2\,dy = 9.9345+120.637=130.572 \approx \underline{131}\text{ cm}^3$

5.13 – Kinematics [AHL]

	SL	HL
A&A	5.9	5.9
A&I		✓

↦ This whole section refers to the study of one-dimensional movement, specifically using calculus. The movement can be described with: *Displacement* (position in relation to the origin), *velocity* (speed, but with a direction, so can be *negative*), and *acceleration*.

↦ We can start by noting that, by definition, velocity is the *rate of change of position* (the clue is in the units, such as metres/sec). Then also, acceleration is the *rate of change of velocity*. What this means is that both velocity & acceleration can also be viewed as *derivatives*. Therefore we get:



→ this means: $v(t)=\frac{ds}{dt}$ & $a(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}$

and also: $\int v(t)\,dt = s(t)+C$ & $\int a(t)\,dt = v(t)+C$

E.G.1 ↦ A helicopter is 1000m high at $t=0$, where $v(t)=-10t$. ⓐ Find $a(t)$: & ⓑ Show that $s(10)=500\text{m}$:

ⓐ → $a(t)=\frac{dv}{dt}=\underline{-10\text{ms}^{-2}}$ // ⓑ → $s(t)=\int v(t)\,dt = -5t^2+C \rightarrow s(0)=-5(0)^2+C \rightarrow s(0)=C=1000$

$\therefore s(t)=-5t^2+1000 \rightarrow s(10)=-5(100)+1000=\underline{500\text{m}}$

E.G.2 ↦ An object moves with displacement $s(t)=t^3-3t-4$ cm, for $t\geq 0$ seconds. Where does it change direction?

→ Must have $v=0$ to change direction → $v=\frac{ds}{dt}=3t^2-3$. Set $3t^2-3=0 \rightarrow 3t^2=3 \rightarrow t^2=1 \rightarrow t=\cancel{-1}$ or 1 sec. Find displacement at $t=1$: $s(1)=(1)^3-3(1)-4=\underline{-6\text{cm}}$. i.e.: 6cm 'to the left.'

DISTANCE TRAVELLED ↦ The distance travelled by an object is clearly closely linked with the concept of *displacement*, so we will be able to *integrate velocity* to help us find the distance travelled. However, dist. travelled takes into account *all* changes in displacement, even *negative* changes (going backwards). You may remember that distance can be found by finding the area under a velocity curve, i.e.: with *definite integrals*, but we *absolute value* to deal with the backwards movement:

In F.B. $\boxed{\text{Dist. Trav.}=\int_{t_1}^{t_2}|v(t)|\,dt}$

↦ If doing this manually, you split up the integrals between their roots – which here are the changes in direction. That way, you can change any 'negative distances' to be positive, just as with area.

E.G.3 ↦ A particle has velocity function, $v(t)=1-2t$ cm/s. Find the distance travelled by the particle in the first second of motion:

→ We need to find any roots to the function first: $1-2t=0 \rightarrow t=0.5$ secs. We could then have a look at the graph of v:

We could split the integral up into $\int_0^{0.5} v\,dt+\int_{0.5}^{1} v\,dt$, but we need to do $\int_0^{0.5} v\,dt+\left|\int_{0.5}^{1} v\,dt\right|$ as the second half of the time, it is going backwards

Really, in AIHL, we should go straight to the GDC and do $\int_0^1 |1-2t|\,dt=\underline{0.5\text{cm}}$

5.14 – Differential Equations [AHL]

	SL	HL
A&A		5.18
A&I		✓

↦ If we take y as a function of x ($y=f(x)$), then we say that a *'differential equation'* is an equation that relates x, y & $\frac{dy}{dx}$ (or higher derivatives), with the general form:

$$\boxed{F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ky}{dx^k}\right)=0}$$ ↳ 'k' is the 'order' of the diff. equation

E.G.1 ↦ $\frac{dy}{dx}+y=x-2$, $y'+y=x-2$, $f'(x)+f(x)=x-2$ → These are three ways of writing the same differential equation. But what do they mean? Well, in words: The derivative of a function of x, plus that same function, is equal to $x-2$.

↦ From this example, you start to see that *'solving'* a differential equation is not a case of just finding a number, but rather finding the unknown *function*: $y=f(x)$ that fits.

↦ Apart a slight exception in 5.18, we only really work with *first-order differential equations*.

FINDING SOLUTIONS ↦ So if the main goal here is to find this function that fits the diff. eq., then we need to know a method for finding it, and need to know when we can use it. A&A HL has a few methods of solving, whereas A&I HL just has one – for *separable equations*.

SEPARABLE EQUATIONS ↦ A *separable equation* is one that is in the form shown here → $\boxed{\frac{dy}{dx}=p(x)\cdot q(x)}$ …or it can at least be rearranged into this form

E.G.2 ↦ Is $x^3y\cdot\frac{dy}{dx}=x-5$ in 'separable form'?
↳ Yes, it can be rearranged to give: $\frac{dy}{dx}=\left(\frac{x-5}{x^3}\right)\cdot\left(\frac{1}{y}\right)$, so $p(x)=\frac{x-5}{x^3}$, $q(y)=\frac{1}{y}$

SOLVING ↦ Now we know when we have the correct form for this, we can discover the steps required to end up with a solution: $y=f(x)$ → the function of x that fits our original differential equation.

1. Write it in the form: $\frac{dy}{dx}=p(x)\cdot q(y)$
2. Rearrange to get $\frac{1}{q(y)}dy=p(x)dx$
3. Integrate both sides, to get $\int\frac{1}{q(y)}dy=\int p(x)dx$ [you only need to put '+c' on one side]
4. Solve for y

E.G.3 ↦ Solve $\frac{dy}{dx}=e^{x-y}$:
↳ Rearrange to $\frac{dy}{dx}=(e^x)(e^{-y})$, then $e^y dy=e^x dx$. Then integrate both sides $\int e^y dy=\int e^x dx \longrightarrow e^y=e^x+c$
$\longrightarrow \underline{y=\ln(e^x+c)}$

E.G.4 ↦ Find the solution of $\frac{dy}{dt}=3t^2y^3+e^ty^3$:
↳ Factor the RHS to get: $\frac{dy}{dt}=y^3(3t^2+e^t)$, rearrange to get → $\frac{1}{y^3}dy=(3t^2+e^t)dt$. Integrate: $\int\frac{1}{y^3}dy=\int(3t^2+e^t)dt$
$\longrightarrow \frac{-1}{2y^2}=t^3+e^t+c \longrightarrow \underline{y=\frac{1}{\sqrt{2(t^3-e^t+c)}}}$

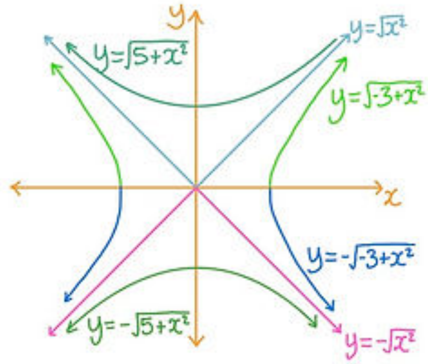
5.15 – Slope Fields [AHL]

	SL	HL
A&A		
A&I		✓

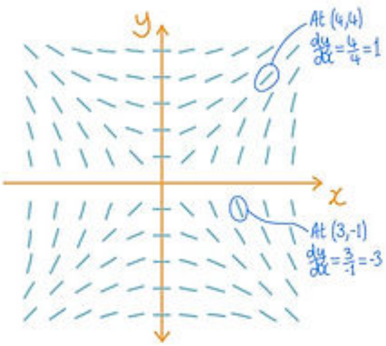
⊩→ In subtopic 5.14, we saw how to solve differential equations to find a solution functions, however, in many examples, we didn't find C. So what we really found were many solutions – a set of solutions. In this section, we try to *graph them*.

E.G.1⊩→ Graph six solution curves for $\frac{dy}{dx} = \frac{x}{y}$

↳ It is separable, so $y\,dy = x\,dx \rightarrow \int y\,dy = \int x\,dx \rightarrow y^2 = x^2 + C$, meaning the solutions are in the form: $y = \pm\sqrt{C + x^2}$, so we could have $y = \pm\sqrt{5+x^2}$, $y = \pm\sqrt{-3+x^2}$, $y = \pm\sqrt{x^2}$. Lets graph those six eqs.:

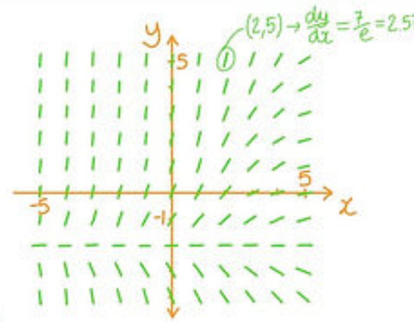


⊩→ This is fine, but is already looking crowded, and how did we even pick which values of C to use? Well, there is an alternative method. We can take a group of points that cover the whole plane, often at all the integer points (or smaller 'round' intervals) – and at each point, we will draw a *short line segment* using the *slope* of a solution curve at that point. These can actually just be found by doing $\frac{dy}{dx}$ at your (x,y) coordinates. If we go back to our opening problem, then at $(6,2)$, $\frac{dy}{dx} = \frac{6}{2} = 3$ or at $(-2,5)$, $\frac{dy}{dx} = \frac{-2}{5}$ or -0.4. Finding all of these and plotting each line creates what we call a *slope field*. ↗



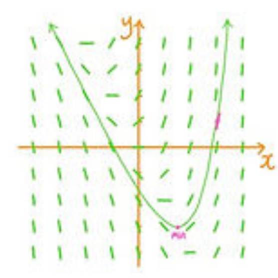
E.G.2⊩→ Draw a slope field for the differential equation of $\frac{dy}{dx} = \frac{y+2}{e^x}$

→ We look for patterns, instead of just finding every single slope. For example, at $y=-2$, $\frac{dy}{dx} = \frac{0}{e^x} = 0$. We call this line of flat slopes: an *isocline*.



NOTE⊩→ You would almost never be asked to draw an entire slope field. You may be asked to analyse one, and potentially *sketch a solution curve*. This is done by taking a point, and *follow the slopes* to make a whole curve. Importantly, this is almost exactly the same as when you might 'find c' after solving a differential equation.

E.G.3⊩→ The slope field shown on the right is for $\frac{dy}{dx} = 2x + y$:



ⓐ→ Find the slope of the tangent to the solution curve at $(3,1)$:

↳ $\frac{dy}{dx} = 2(3) + 1 = 7$. Shown in pink /.

ⓑ→ Sketch the particular solution passing through $(3,1)$:

↳ Extend both ends from the $(3,1)$ line segment

ⓒ→ Estimate a minimum point for the solution curve:

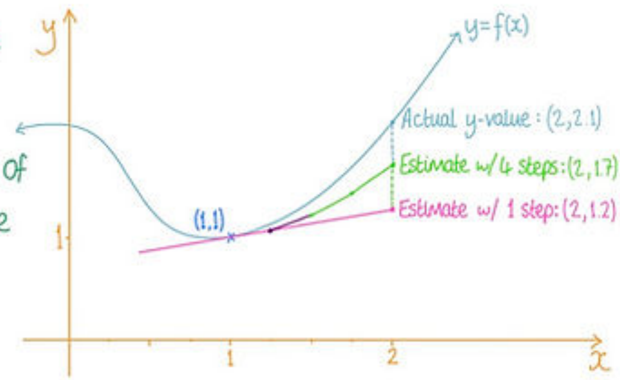
↳ Seems to be around $y=-3$. To be a minimum, $\frac{dy}{dx} = 0 \rightarrow 2x + y = 0 \rightarrow 2x - 3 = 0 \rightarrow x = 1.5 \rightarrow \cdot \underline{(1.5, -3)}$

5.16 – Euler's Method [AHL]

	SL	HL
A&A		(5.18)
A&I		✓

GOAL ↦ In this section, we are not trying to 'solve' a differential equation. Instead, here we are just trying to approximate a y-coordinate of a point on the solution curve, given an initial point, and the differential equation.

THEORY ↦ With the differential equation, we can use it for one of its fundamental purposes – to find slopes of tangents. We find the slope of the tangent at our initial point, and move along that line towards our target x-value. At certain points, we can readjust the slope/direction, to attempt to follow our unknown solution curve to some level of accuracy. Eventually we reach our target x-value, and whatever y-value we have at that point is our estimate.



DEMO ↦ The diagram on the right gives a general sense of how Euler's Method works. Note: you wouldn't be able to see the solution curve whilst doing this. (1,1) is our initial point. The pink line shows the tangent of the curve at (1,1). If we move 1 unit right along the pink line, we reach (2,~1.2). So 1.2 is our estimate for y(2) using 1 'step'. You can see it is not very accurate. If instead we stop $x=1.25$ (point ·), and $\frac{dy}{dx}$ at $x=1.25$, we can then move in that direction for a short time, as that is certain to keep us closer to the solution curve than the pink line. If we do the same slope adjustment at $x=1.5$ & 1.75, then the y-coordinate, ~1.7, is our 4-step estimate of y(2) using Euler's method. It is still not wildly accurate, but if we kept adding steps (shortening step length), accuracy increases.

FORMAL METHOD ↦ To sum up this iterative process to find this y-value estimate formally:

> For a diff. eq. $\frac{dy}{dx}=F(x,y)$ and initial condition $y(x_0)=y_0$. Given n^{th} point (x_n,y_n) & step size $=h$, $(n+1)^{th}$ point is given by $x_{n+1}=x_n+h$ & $y_{n+1}=y_n+h\cdot F(x_n,y_n)$

E.G.1 ↦ Given $\frac{dy}{dx}=x^2+y^2$, and initial point (0,1), using a step length of 0.2, estimate y when $x=1$:

↳ MANUALLY: Calculate $F(x_n,y_n)$ for each step, × by 0.2, + to y_n to get y_{n+1}, until we have $x=1$:

$x_1=0+0.2=0.2$ / $y_1=1+0.2(0^2+1^2)=1.2$

$x_2=0.2+0.2=0.4$ / $y_2=1.2+0.2(0.2^2+1.2^2)=1.496$

$x_3=0.6$ / $y_3=1.496+0.2(0.4^2+1.496^2)=1.9756$

$x_4=0.8$ / $y_4=1.9756+0.2(0.6^2+1.9756^2)=2.8282$

$x_5=1$ / $y_5=2.8282+0.2(0.8^2+2.8282^2)=\underline{4.556}$

GDC: TI-nspire ↦ Lists & Spreadsheets:

	x	y	dy/dx
1	0	1	$=a_1^2+b_1^2$
2	$=a_1+0.2$	$=b_1+c_1$	
3			
4			
5			
6	1	4.556	

$F(x_0,y_0)$

Drag down formulae

solution

TI-84 ↦ Just do the process on the left using the GDC

5.16 continued – Euler's Method for Coupled Systems

COUPLED SYSTEMS ↦ This is similar to related rates, but only in sense that we'll look at the relationship between more than two variables – usually x, y, & *time* (t). Specifically, it will be a *system of 2 differential equations*, one showing how x *varies over time*: $\frac{dx}{dt}$ and how y *varies over time*: $\frac{dy}{dt}$. Both of these can be functions of x, y & t.

USES ↦ Whilst this can be presented abstractly, it can have many interesting real-world *applications*. The main one being *predator-prey* models – how species A & B both differ over time, in relation to the two pop^{n} numbers. Other uses include: voltage/current systems, harmonic motion and oscillators.

NOTATION ↦ We could say either this: $\begin{cases} \frac{dx}{dt} = 5x - 2xy \\ \frac{dy}{dt} = xy - 3y \end{cases}$ or this: $\begin{cases} \dot{x} = 5x - 2xy \\ \dot{y} = xy - 3y \end{cases}$

EULER'S METHOD ↦ Whilst we will consider *solutions* to these systems in 5.17, for now, we will just see how to find *approximate numerical solutions* using Euler's Method. It will work in a very similar way to before:

- You will be given an initial x & y value, but also an initial 't' – which is new.
- The goal will be to *estimate* x & y at a future point in time.
- You will be given a step length again, but here it will be in small periods of *time*.
- $\frac{dx}{dt}$ & $\frac{dy}{dt}$ tell you the rate of change of x & y, so we calculate it for our initial point, and move *at that rate* for a certain short time period.
- After moving for that time period, the rates will have changed. So we recalculate at the next point, and move at those new rates for another step. Repeat this process until the final point in time, you have an x & y estimate.

GDC ↦ You will use the *formulae* in *Lists & Spreadsheets* on the TI-nspire. On the TI-84, you can change the mode from 'FUNC' to 'SEQ' and type in appropriate formulas for x & y into u_n & v_n. Whatever the case, the formulae are:

If $\frac{dx}{dt} = f_1(x,y,t)$ & $\frac{dy}{dt} = f_2(x,y,t)$: $\boxed{t_{n+1} = t_n + h \;//\; x_{n+1} = x_n + h \cdot f_1(x_n, y_n, t_n) \;//\; y_{n+1} = y_n + h \cdot f_2(x_n, y_n, t_n)}$ → and h is the step length

---

E.G.2 ↦ The population of moose (m) and wolves (w) on an island, after t years, are modelled by the coupled system: $\begin{cases} \frac{dm}{dt} = \frac{1}{200} m(m - 20 - 3w) \\ \frac{dw}{dt} = \frac{1}{200} w(3m - 300 - 5w) \end{cases}$ Initially, there are 150 moose and 40 wolves. Using a step length of $t = 0.1$, estimate the amount of moose & wolves on the island after a year:

↳ MANUALLY: $m_0 = 150$ & $w_0 = 40$. We can calculate rates at $t = 0$:

$\frac{dm}{dt} = \frac{150}{200}(150 - 20 - 3(40)) = \frac{3}{4}(10) = 7.5$

$\frac{dw}{dt} = \frac{40}{200}(3(150) - 300 - 5(40)) = \frac{1}{5}(-50) = -10$

So $m_1 = 150 + 0.1(7.5) = 150.75$

and $w_1 = 40 + 0.1(-10) = 39$

So after $\frac{1}{10}$ of a year, our numbers are $(150.75, 39)$

We'd have to do this 9 more times to get our answer!!

So we use our GDC instead →→

GDC: TI-nspire ↦ Lists & Spreadsheets:

	t	m	w
1	0	150	40
2	$=a_1+0.1$	$=b_1+0.1\left(\frac{b_1}{200}\right)(b_1-20-3c_1)$	$=c_1+0.1\left(\frac{c_1}{200}\right)(3b_1-300-5c_1)$
3			
4			
⋮			
10	1	169.492	36.741

Drag down formulae

solutions

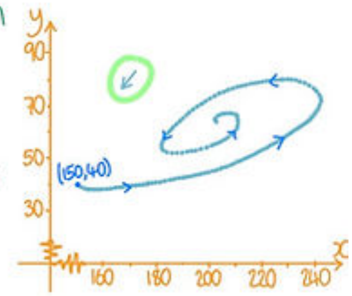
So we round to ≈ 169 moose & 37 wolves

5.17 – Phase Portraits & Solutions [AHL]

	SL	HL
A&A		
A&I		✓

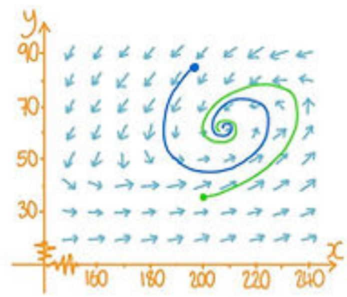
COUPLED LINEAR SYSTEMS ↦ The coupled diff. eqs. in this section will be in the form: $\begin{cases}\dot{x}=ax+by\\ \dot{y}=cx+dy\end{cases}$

VISUALISING SOLUTIONS ↦ If we take the Euler's Method approximations for E.G.2 from 5.16, and extend it down for 10 years instead, and plot the x & y [m & w] values, then we get a good visualisation of the populations over time, given the (150,40) starting point:



We could also just look at slope(s) at certain points, such as (170, 80), also shown on the graph: $\frac{dx}{dt}=\frac{1}{200}x(x-20-3y)=\frac{1}{200}(170)(170-20-3(80))\approx-65$ & $\frac{dy}{dt}\approx-76$. This tells us that if the pop^n's are ever 170 & 80, then both will decrease from there. This is why we have an arrow facing down & left ↙.

PHASE PORTRAITS ↦ You could have arrows covering the entire coordinate plane, and it would show the general shape of any *solution curve*, showing the *trajectories* they would take. This means you could take any initial point, and follow the arrows to estimate a solution curve. For example, taking the same situation, but imagine we start with (200,35) or (196, 85). The lines shown from • & • simply follow the arrows, and they all seem to *spiral* in on what we call an *equilibrium point*. This means in the long-term, given any of three initial populations mentioned so far, the eventual numbers would be ~210 & ~60. Overall, this graph is called a *phase portrait*.



EQUILIBRIA ↦ The curves won't always just centre on a single point, and they won't always follow this type of spiral either. There are six types of equilibrium point of phase portraits for our linear coupled diff. eqs.:

STABLE FIXED — UNSTABLE FIXED — STABLE SPIRAL — UNSTABLE SPIRAL — SADDLE — CENTRE

So for example, the phase portrait above has a stable spiral at ~(210,60).

SOLUTIONS ↦ We were able to solve regular single diff. eqs. if they were separable. The format of the 'solution' then was finding y: a function of x. So what is the *format* of the *solution* for one of these coupled systems? They are *vectors*: ones where both x & y components are functions of t. Without any extra info, there are infinitely many solution curves. If you are given an initial point, you can find a specific solution, just like tracing one curve on the phase portrait. If we write a coupled system in *matrix form*, then it can be used to find the *general solution*:

If $\left.\begin{matrix}\dot{x}=ax+by\\ \dot{y}=cx+dy\end{matrix}\right\}$ then the *general solution* is $\boxed{\vec{x}=Ae^{\lambda_1 t}\vec{p}_1+Be^{\lambda_2 t}\vec{p}_2}$ → In F.B.

...where λ_1 & λ_2 are the *eigenvalues* of $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ and $\vec{p}_1$ & $\vec{p}_2$ are the *eigenvectors* ← (Not clarified in F.B.)

5.17 continued

E.G.1 ↦ Take the coupled system $\begin{rcases}\dot{x}=3x+y \\ \dot{y}=4x+3y\end{rcases}$. (a) Find a general solution:

↳ We take $\begin{pmatrix}3&1\\4&3\end{pmatrix}$ and find eigenvalues: $\left|\begin{pmatrix}\lambda-3&-1\\-4&\lambda-3\end{pmatrix}\right|=0 \rightarrow (\lambda-3)^2-(-1)(-4)=0 \rightarrow \lambda_1=5$ & $\lambda_2=1$

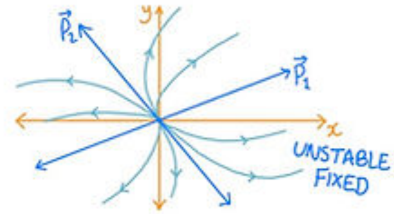
Find eigenvectors: $\left(\begin{pmatrix}5&0\\0&5\end{pmatrix}-\begin{pmatrix}3&1\\4&3\end{pmatrix}\right)\begin{pmatrix}a\\b\end{pmatrix}=0 \rightarrow 2a-b=0 \rightarrow \vec{p}_1=\begin{pmatrix}1\\2\end{pmatrix}$. // $\vec{p}_2=\begin{pmatrix}1\\-2\end{pmatrix}$ $\therefore \begin{pmatrix}x\\y\end{pmatrix}=Ae^{5t}\begin{pmatrix}1\\2\end{pmatrix}+Be^{t}\begin{pmatrix}1\\-2\end{pmatrix}$

(b) Given that a particular solution curve passes through $(-1,-14)$ at $t=0$, find the solution:

↳ $\begin{pmatrix}x\\y\end{pmatrix}=Ae^{5(0)}\begin{pmatrix}1\\2\end{pmatrix}+Be^{(0)}\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}-1\\-14\end{pmatrix} \rightarrow A\begin{pmatrix}1\\2\end{pmatrix}+B\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}-1\\-14\end{pmatrix} \rightarrow \begin{cases}A+B=-1\\2A-2B=-14\end{cases} \rightarrow A=-4, B=3 \rightarrow \begin{pmatrix}x\\y\end{pmatrix}=-4e^{5t}\begin{pmatrix}1\\2\end{pmatrix}+3e^{t}\begin{pmatrix}1\\-2\end{pmatrix}$

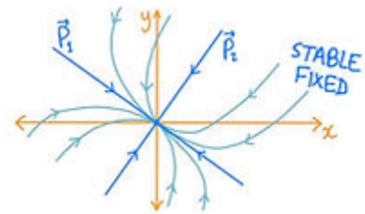
DETERMINING EQUILIBRIUM TYPES ↦ So we can do better than just identifying the type of equilibrium point by looking at a phase portrait. Now, we will actually be able to identify the type directly from the diff. eqs. themselves. Specifically, the format of the eigenvalues will determine the type. Eigenvectors help with direction. If eigenvalues are....

→ Both real & positive: where $\lambda_1>\lambda_2>0$, then it is unstable fixed, w/ trajectories starting along $\vec{p}_2$, then tend towards the $\vec{p}_1$ direction.



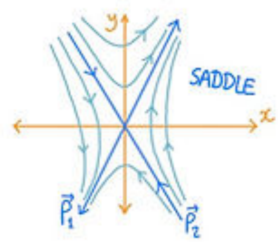
→ Complex, positive real parts: then it is an unstable spiral.

→ Both real & negative: where $0>\lambda_1>\lambda_2$, then it is stable fixed, w/ trajectories approaching (0,0) along $\vec{p}_1$, starting parallel to $\vec{p}_2$.



→ Complex, real parts = 0: then it is a centre.

→ Real, 1 positive/1 negative: where $\lambda_1>0>\lambda_2$, then it is a saddle, w/ trajectories tending towards the $\vec{p}_1$ direction.



↳ Complex, negative real parts: then it is a stable spiral.

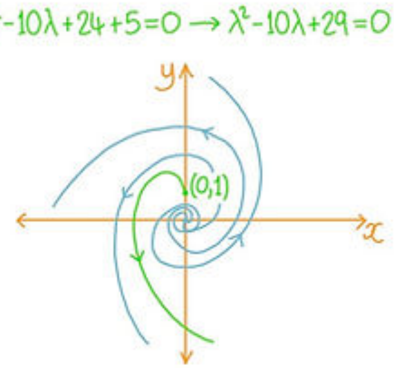
NOTE ↦ The equilibrium point will be at (0,0) for these linear systems.

DIRECTION OF SPIRAL ↦ Pick a point, find $\frac{dx}{dt}$ & $\frac{dy}{dt}$ at that point, then the direction tells you: cl.-wise or anticl.-wise.

E.G.2 ↦ Sketch the phase portrait for $\begin{rcases}\frac{dx}{dt}=6x-y\\ \frac{dy}{dt}=5x+4y\end{rcases}$, showing the trajectory from (0,1):

↳ Find characteristic polynomial of $\begin{pmatrix}6&-1\\5&4\end{pmatrix} \rightarrow (\lambda-6)(\lambda-4)-(-1)(5)=0 \rightarrow \lambda^2-10\lambda+24+5=0 \rightarrow \lambda^2-10\lambda+29=0$

Solve: $\lambda=\frac{10\pm\sqrt{-16}}{2}=5\pm2i$. Complex, with positive real parts. Therefore, we get an unstable spiral. For (0,1), we can find its initial slope $\rightarrow \frac{dx}{dt}=6(0)-1=-1$ // $\frac{dy}{dt}=5(0)+4(1)=4$, so this means above the equilibrium point, it is going up & left, and for this position, its means it rotates anti-clockwise.



5.18 – Second-Order Euler's Method [AHL]

	SL	HL
A&A		
A&I		✓

SECOND-ORDER DIFF. EQS. ⊩ A *second-order differential equation* is simply an equation that can be written in the form: $\frac{d^2x}{dt^2}=f(x,\frac{dx}{dt},t)$. This may be daunting, but is just an equation that links our two variables, the first derivative, and crucially: *the second derivative* too, which we haven't seen before. This could be something like $32\frac{d^2x}{dt^2}-\frac{dx}{dt}+4x=t$ or even $2u''-7u'=u$.

SUBSTITUTION ⊩ The main technique here will always begin by *converting it to a coupled system*. You'll be able to achieve this by *substituting* $y=\frac{dx}{dt}$. This can also be differentiated $\rightarrow \frac{d}{dt}(y)=\frac{d}{dt}\left(\frac{dx}{dt}\right) \rightarrow \frac{dy}{dt}=\frac{d^2x}{dt^2}$ → which is the second substitution, giving you $\frac{dy}{dt}=\ldots$, meaning you have both elements of a *coupled system* of *diff. eqs.*.

EULER'S METHOD ⊩ So once you convert, you can carry out Euler's Method just like in 5.16-pg.2:

E.G.1 ⊩ Given $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+3x=\sin(t)$, and at $t=0$, $x=1$ and $\frac{dx}{dt}=-1$, estimate x at $t=1$, given a step length of 0.2:

↳ Let $y=\frac{dx}{dt}$ (and $\frac{dy}{dt}=\frac{d^2x}{dt^2}$), this gives us $\frac{dy}{dt}+2y+3x=\sin(t)$. This means we have: $\begin{cases}\dot{x}=y\\ \dot{y}=\sin(t)-3x-2y\end{cases}$

Our GDC gives us the table:

t	x	$y(=\dot{x})$
0	1	-1
0.2	0.8	-1.2
⋮	⋮	⋮
1	0.00581	-0.333

∴ our answer is $x(1)=\underline{0.00581}$

LINEAR ⊩ To be able answer more interesting problems, the coupled system would need to be in our *'linear'* form, i.e.: $\left.\begin{matrix}\dot{x}=ax+by\\ \dot{y}=cx+dy\end{matrix}\right\}$. For this to happen, you would have to be given: $\frac{d^2x}{dt^2}+m\frac{dx}{dt}+nx=0$, and will always convert to get $\begin{cases}\dot{x}=y\\ \dot{y}=-nx-my\end{cases}$, which technically in our desired form.

E.G.2 ⊩ ⓐ → For the diff. eq. $2\frac{d^2x}{dt^2}+5\frac{dx}{dt}=3x$, given $x(0)=-4$ & $\left.\frac{dx}{dt}\right|_{t=0}=5$, estimate $x(2)$ using E.M. with $h=0.5$:

↳ ⓐ → As usual, we use $y=\frac{dx}{dt}$ & $\frac{dy}{dt}=\frac{d^2x}{dt^2}$, so $2\left(\frac{dy}{dt}\right)+5y=3x \longrightarrow \frac{dy}{dt}=\frac{3}{2}x-\frac{5}{2}y$. This, along with the equation $\frac{dx}{dt}=y$ is our coupled system. We then use the GDC to find $x(2)$. In C_2, we type in $=C_1+0.5(1.5b_1+2.5C_1)$ and in b_2, we type $=b_1+0.5(C_1)$. Drag it down to $t=2$, and we get $x(2)=-5.00781\approx\underline{-5.01}$

t	x	y
0	-4	5
0.5	-1.5	-4.25
1	-3.625	-0.0625
1.5	-3.65625	-2.70313
2	-5.00781	-2.06641

⊢ ⓑ → For the same initial conditions, find a particular solution for $2\frac{d^2x}{dt^2}+5\frac{dx}{dt}=3x$, and use it to find the exact value of $x=2$:

↳ Again, we want to use our coupled system: $\begin{cases}\dot{x}=y\\ \dot{y}=1.5x-2.5y\end{cases} \rightarrow$ solve $\det\begin{pmatrix}\lambda-0 & -1\\ -1.5 & \lambda+2.5\end{pmatrix}=0 \rightarrow \lambda(\lambda+2.5)-(-1)(-1.5)=0$

$\rightarrow \lambda^2+2.5\lambda-1.5=0 \longrightarrow \lambda_1=\frac{1}{2}$ & $\lambda_2=-3$

$\vec{p}_1=\left(\begin{pmatrix}0.5 & 0\\ 0 & 0.5\end{pmatrix}-\begin{pmatrix}0 & 1\\ 1.5 & -2.5\end{pmatrix}\right)\begin{pmatrix}a\\ b\end{pmatrix}=0 \rightarrow \begin{pmatrix}0.5 & -1\\ -1.5 & 3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=0 \rightarrow 0.5a-b=0 \longrightarrow \vec{p}_1=\begin{pmatrix}2\\ 1\end{pmatrix}$ Similarly $\lambda_2=-3 \rightarrow \vec{p}_2=\begin{pmatrix}1\\ -3\end{pmatrix}$

$\begin{pmatrix}x\\ y\end{pmatrix}=Ae^{\frac{1}{2}t}\begin{pmatrix}2\\ 1\end{pmatrix}+Be^{-3t}\begin{pmatrix}1\\ -3\end{pmatrix} \xrightarrow[x=-4,\,y=5]{\text{using } t=0} Ae^{0}\begin{pmatrix}2\\ 1\end{pmatrix}+Be^{0}\begin{pmatrix}1\\ -3\end{pmatrix}=\begin{pmatrix}-4\\ 5\end{pmatrix} \rightarrow \begin{cases}2A+B=-4\\ A-3B=5\end{cases} \rightarrow A=-1$ & $B=-2$ $\therefore \begin{pmatrix}x\\ y\end{pmatrix}=-e^{\frac{1}{2}t}\begin{pmatrix}2\\ 1\end{pmatrix}-2e^{-3t}\begin{pmatrix}1\\ -3\end{pmatrix}$

x component is $x=-2e^{\frac{1}{2}t}-2e^{-3t}$, at $t=2$: $x=-2e^{\frac{1}{2}(2)}-2e^{-3(2)}=-2e-\frac{2}{e^6}=-5.43656\ldots-0.00496\ldots\approx\underline{-5.44}$