

TOPIC 3 TRIG. & GEOMETRY

	SL	HL
ALA	✓	✓
AKI	✓	✓

3.1 - Basics

→ In the previous course, everything in this section was assumed prior knowledge - because of that, I will just present these formulae, w/o deriving.

DISTANCE BETWEEN POINTS → This is found by creating a right-angled triangle and using the *pythagorean theorem* to find the distance (the *hypotenuse*):

2D: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

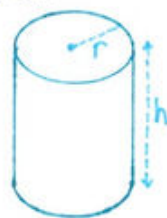
3D: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

CYLINDER → "Circular - prism"

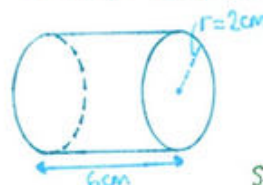
$Vol. = \pi r^2 h$

$S.A. = 2\pi r h + 2\pi r^2$

Curved face



E.G. 1 → Find vol. & S.A.:



$Vol. = \pi (2)^2 \cdot 6$
 $= 24\pi \text{ cm}^3$
 or 75.4 cm^3

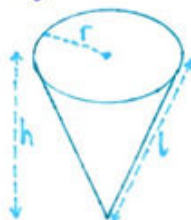
$S.A. = 2\pi (2)^2 + 2\pi (2)(6) \approx 101 \text{ cm}^2$

CONE → "Circular-based pyramid"

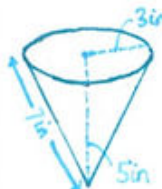
$Vol. = \frac{1}{3}(\pi r^2 h)$

$S.A. = \pi r l + \pi r^2$

curved face



E.G. 2 → Find vol. & S.A.:



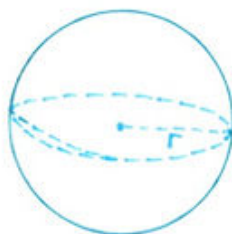
$Vol. = \frac{\pi (3)^2 \cdot 4}{3} = \frac{\pi \cdot 36}{3}$
 $= 12\pi \text{ in}^3$ or 37.7 in^3

$S.A. = \pi (3)(5) + \pi (3)^2$
 $= 15\pi \text{ in}^2$ or 47.1 in^2

SPHERE →

$Vol. = \frac{4}{3}(\pi r^3)$

$S.A. = 4\pi r^2$



E.G. 3 → Find Vol. & S.A.:



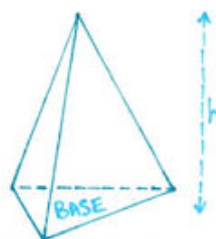
$Vol. = \frac{4\pi (9)^3}{3} = \frac{4\pi \cdot 729}{3}$
 $= 972\pi \text{ m}^3$
 or 3052 m^3

$S.A. = 4\pi (9)^2 = 324\pi \text{ m}^2$
 or 1017 m^2

PYRAMID →

$Vol. = \frac{BASE AREA \times h}{3}$

S.A. → No formula
 → Just add areas of all faces



E.G. 4 → Find V. & S.A.:



$Vol. = \frac{B \cdot h}{3}$
 $= \frac{(6 \times 6) \cdot 10}{3} = 120 \text{ cm}^3$

$S.A. = (4 \times \text{Triangle}) + \text{Square}$
 $= (4 \times \frac{6 \times 12}{2}) + (6 \times 6)$
 $= 180 \text{ cm}^2$

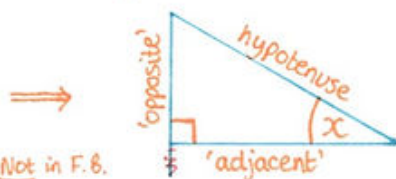
3.2 - Triangles

	SL	HL
AA	✓	✓
AI	✓	✓

TRIG. RATIOS → One may notice that when you have a right-angled triangle, and one other known angle, it doesn't matter what 'size' the triangle is, the *ratios* between the three sides remain constant. The ratios can be calculated with trigonometric functions: sine, cosine & tangent [sin, cos, tan].

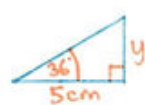
→ Each one of these relates to a ratio between a pair of sides:

S O H	C A H	T O A
$\downarrow \sin x = \frac{\text{Opp.}}{\text{Hyp.}}$	$\downarrow \cos x = \frac{\text{Adj.}}{\text{Hyp.}}$	$\downarrow \tan x = \frac{\text{Opp.}}{\text{Adj.}}$



← Not in F.B.

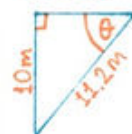
E.G. 1 → Find y : We have opp. & adj. ∴ use tan:



$$\tan x = \frac{\text{Opp.}}{\text{Adj.}} \rightarrow \tan(36^\circ) = \frac{y}{5}$$

$$y = 5 \cdot \tan(36^\circ) \rightarrow y = \underline{3.63\text{cm}}$$

E.G. 2 → Find θ : We have opp. & hyp. ∴ use sine:



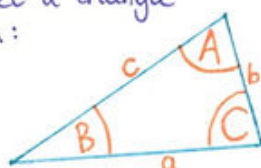
$$\sin \theta = \frac{10}{11.2} \rightarrow \theta = \sin^{-1}\left(\frac{10}{11.2}\right)$$

'Inverse sine'

$$\theta = \underline{63.2^\circ}$$

ALL TRIANGLES → As any triangle can be split into two right-angled triangles, these rules may be extended to let us work with any type of triangle:

If we label a triangle as such:



... then we have the following:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{'sine rule'}$$

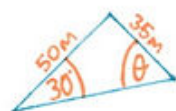
ALL in F.B.

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \quad \text{'cosine rule'}$$

$$\text{Area} = \frac{1}{2} ab \sin C \quad \text{'Area rule'}$$

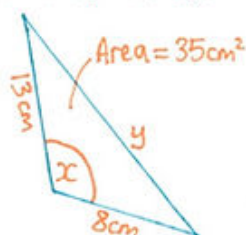
E.G. 3 → Find θ : We have pairs of opposite angles & sides ∴ Use sine rule:



$$\frac{\sin \theta}{50} = \frac{\sin(30^\circ)}{35} \rightarrow \sin \theta = \frac{50 \sin(30^\circ)}{35}$$

$$\theta = \sin^{-1}\left(\frac{5}{7}\right) \rightarrow \theta = \underline{45.6^\circ}$$

E.G. 4 → Find x : We are given the area, so try the area rule.



$$\text{Area} = \frac{1}{2} ab \sin C$$

$$35 = \frac{1}{2} (13)(8) \sin C$$

$$\sin C = \frac{35}{52} \rightarrow C = \sin^{-1}\left(\frac{35}{52}\right)$$

$$C = \underline{42.3^\circ}, \underline{138^\circ}$$

E.G. 5 → Find side y from E.G. 4:

→ We are considering three sides and an angle, so we use the cosine rule:

$$y^2 = 8^2 + 13^2 - 2(8)(13) \cos(135^\circ)$$

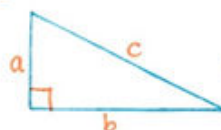
$$y^2 = 64 + 169 - 208 \cos(135^\circ)$$

$$y = \sqrt{380.078...} \rightarrow y = \underline{19.5\text{cm}}$$

3.3 - Trig. Applications

	SL	HL
A&A	✓	✓
A&I	✓	✓

→ There are many possibilities for real-world situations in which you may have to use the trig. rules from 3.2, or SOHCAHTOA, or even the Pythagorean Theorem:



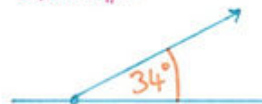
$$\Rightarrow a^2 + b^2 = c^2 \quad \text{OR} \quad \begin{matrix} c^2 - a^2 = b^2 \\ c^2 - b^2 = a^2 \end{matrix}$$

NOT IN F.B.

→ There are a couple of pieces of terminology you may need to know:

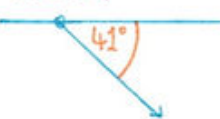
ELEVATION/DEPRESSION → The angle of elevation/depression refers to the angle between the observer's line of sight, and the horizontal.

E.G. 1 →



Angle of elevation is 34°

E.G. 2 →



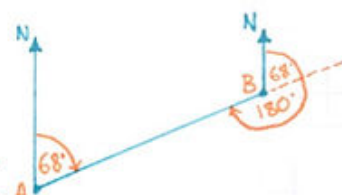
Angle of depression is 41°

BEARINGS → This is used in navigation to describe a direction from one point to another, measured clockwise from North.

E.G. 3 →

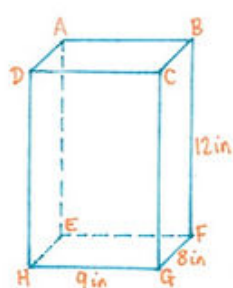
A to B: 068°

B to A: $68 + 180 = \underline{248°}$



REAL-WORLD APPLICATIONS → A couple of tips → always consider a diagram, and if you are stuck, finding the size of any angle or side length possible is always a good place to start.

E.G. 4 → Find angle $\angle FHB$

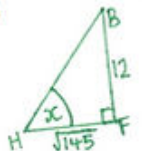


Find FH: $FH^2 = 8^2 + 9^2$
 $FH = \sqrt{145}$

Consider $\triangle FHB$:

$$\tan x = \frac{12}{\sqrt{145}}$$

$$x \approx 44.9°$$



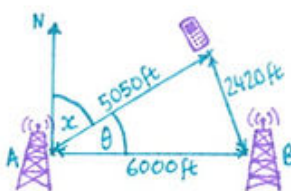
E.G. 5 → Find the bearing from mast A to the phone, given mast B is due East of A:

Find θ with cosine rule:

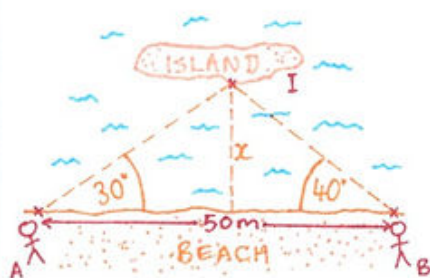
$$\cos \theta = \frac{5050^2 + 6000^2 - 2420^2}{2(5050)(6000)}$$

$$\theta = \cos^{-1}(0.918...) = 23.2°$$

$$\text{Bearing } x = 90 - 23.2 = 066.8°$$



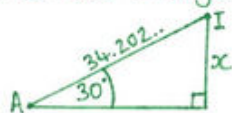
E.G. 6 → Find x , the shortest distance to from the beach to the island:



Find angle $\angle AIB$: $180 - 40 - 30 = 110°$

Use sine rule to find AI: $\frac{AI}{\sin 40} = \frac{50}{\sin 110}$, $AI = 34.202...m$

Make new triangle:



$$\sin 30 = \frac{x}{34.202}, \quad x = \underline{17.1m} \text{ to swim}$$

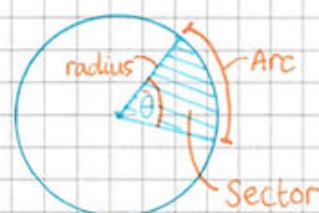
3.4 - Arc/Sector

	SL	HL
A&A		
A&I	✓	✓

RADIANS → You have learned about, or heard of, *radians* as another measure for angles. It is *NOT* in A&I SL (it is in all 3 other courses).

CIRCLES → Here we remember some facts and terminology about circles.

$$\boxed{\text{Area} = \pi r^2} \quad \boxed{\text{Circumference} = 2\pi r}$$



LENGTH OF ARCS → Looking at the diagram above, it should be clear that an arc is simply a fraction of the circumference. In fact, we can view it as a fraction of the 360° that makes up a circle, specifically: $\frac{\theta}{360}$

$$\therefore \boxed{\text{Arc Length} = \frac{\theta}{360} \times 2\pi r} \rightarrow \text{In F.B.}$$

SECTOR AREA → We have a similar situation here, and it is again, the $\frac{\theta}{360}$ fraction of the area.

$$\therefore \boxed{\text{Sector Area} = \frac{\theta}{360} \times \pi r^2} \rightarrow \text{In F.B.}$$

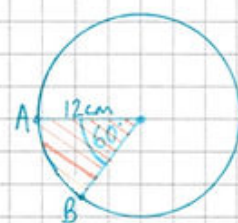
E.G. 1 → For the diagram shown:

a) Find the shaded area:

$$\hookrightarrow \frac{60}{360} \times \pi \times 12^2 = \frac{1}{6} \times \pi \times 144 = \underline{24\pi \text{ cm}^2 / 75.9 \text{ cm}^2}$$

b) Find the minor arc length, AB:

$$\hookrightarrow \frac{60}{360} \times 2\pi \times 12 = \frac{1}{6} \times \pi \times 24 = \underline{4\pi \text{ cm} / 12.6 \text{ cm}}$$



E.G. 2 → Find the perimeter of shaded region:

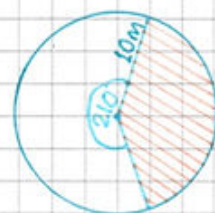
→ Firstly, find angle of shaded region:

$$360^\circ - 210^\circ = 150^\circ$$

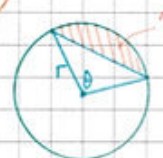
$$\text{Arc length} = \frac{150}{360} \times 2\pi r = \frac{5}{12} \times 20 \times \pi$$

$$\approx 26.2 \text{ m}$$

$$\text{Perimeter} = 10 + 10 + 26.2 = \underline{46.2 \text{ m}}$$



→ Be prepared to use these rules along with trig. triangle rules:

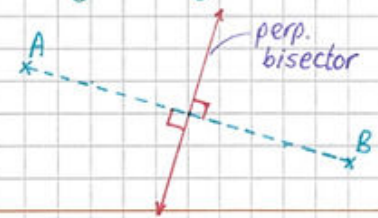


3.5 - Perpendicular Bisectors

	SL	HL
A&B		
A&C	✓	✓

PERPENDICULAR BISECTORS → A bisector is a line that cuts halfway between two points, and as you know, perpendicular is 'at right-angles'.

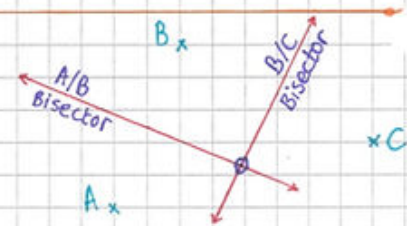
Putting those together, the perp. bisector is perp. to the line segment between A&B, and passes through the midpoint of A&B.



PROPERTIES → These have certain properties:

- Every point on the perpendicular bisector is equidistant from A&B.
- Every point left of the line is closer to A, and vice versa.

3-POINTS → If you have 3 points, all you need is two (or three) perp. bisectors, and their intersection point is equidistant from A, B & C, all at once:



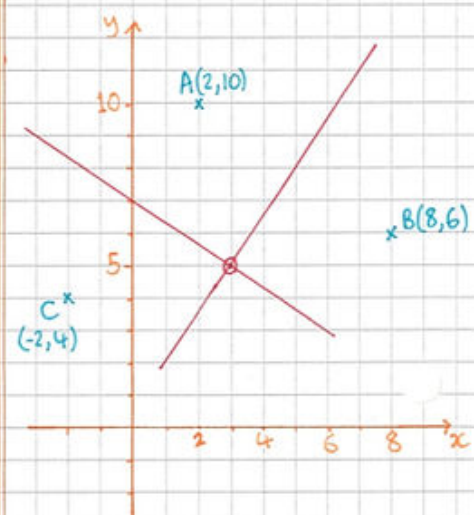
EQUATION → You can find the equation of a perp. bisector, because we can...

- ...find the slope between two points
- ...calculate a slope of a line perpendicular to that
- ...find the midpoint of two points
- ...find the equation of the bisector, given its slope, and one point on it (the midpoint) → either with a $y = mx + c$ or $(y - y_1) = m(x - x_1)$ method.

E.G. 1 → Find the intersection point of the perpendicular bisectors of A&B and of A&C:

↳ Firstly, the P.B. of A&B: Slope between A&B → $\frac{10-6}{2-8} = \frac{4}{-6} = -\frac{2}{3}$. Perp. to that = $\frac{3}{2}$. Midpoint = $(\frac{2+8}{2}, \frac{10+6}{2}) = (5, 8)$. $y - y_1 = m(x - x_1) \rightarrow y - 8 = \frac{3}{2}(x - 5) \rightarrow y = \frac{3}{2}x + \frac{1}{2}$ // Same for A&C gives $y = -\frac{2}{3}x + 7$.

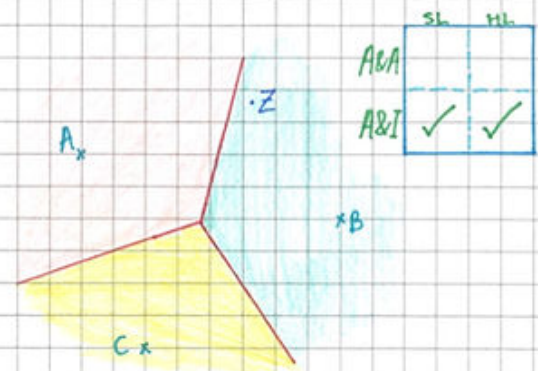
Int. point: $\frac{3}{2}x + \frac{1}{2} = -\frac{2}{3}x + 7 \rightarrow \frac{9}{6}x + \frac{3}{6} = -\frac{4}{6}x + \frac{42}{6} \rightarrow 9x + 3 = -4x + 42 \rightarrow 13x = 39 \rightarrow x = 3$. Plug in: $y = -\frac{2}{3}(3) + 7 \rightarrow y = 5$. Hence: (3, 5)



3.6 - Voronoi Diagrams

→ We have already seen how, with 3 points, that centre point is equidistant. The other property here though, is that every point in each region, or *cell*, is closest

to the x , or *site*, in that cell. i.e.: If we take $\cdot Z$, and want to know: which site it is closest to? We see it is in B's cell, so it is closest to B.



VORONOI DIAGRAM → A Voronoi diagram extends this to many points. The same principles still apply, with every point in each *cell* being closest to the *site* in that cell. The example below shows a Voronoi diagram, and asks some questions that might be typical:

E.G. 1 → For this Voronoi diagram:

a) Give coordinates of a vertex that is equidistant from A, B & F → (3, 8)

b) (3, 4) is closest to which site?

→ Site D

c) Give coordinates of any point equidistant from B & C → (9, 6) [or (7, 10)]

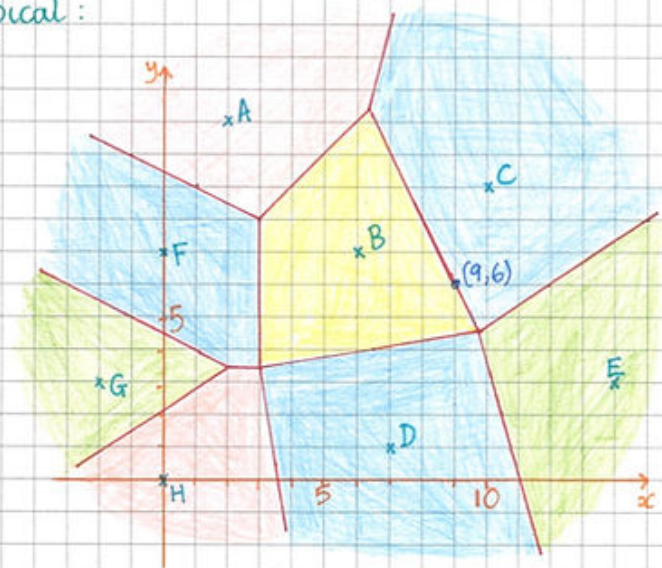
d) Which site has the smallest cell?

→ It may not look obvious at this scale, but every cell is infinite, apart from B, which has *edges* on all sides.

USES → These are used in the real-world for forestry - helping predict forest fires, in chemistry - on a molecular level, and one of the most common: cell phone towers - determining which tower anyone is closest to.

→ You can expect any of the questions above, as well as constructing a very small Voronoi diagram, and adding a site to an existing diagram.

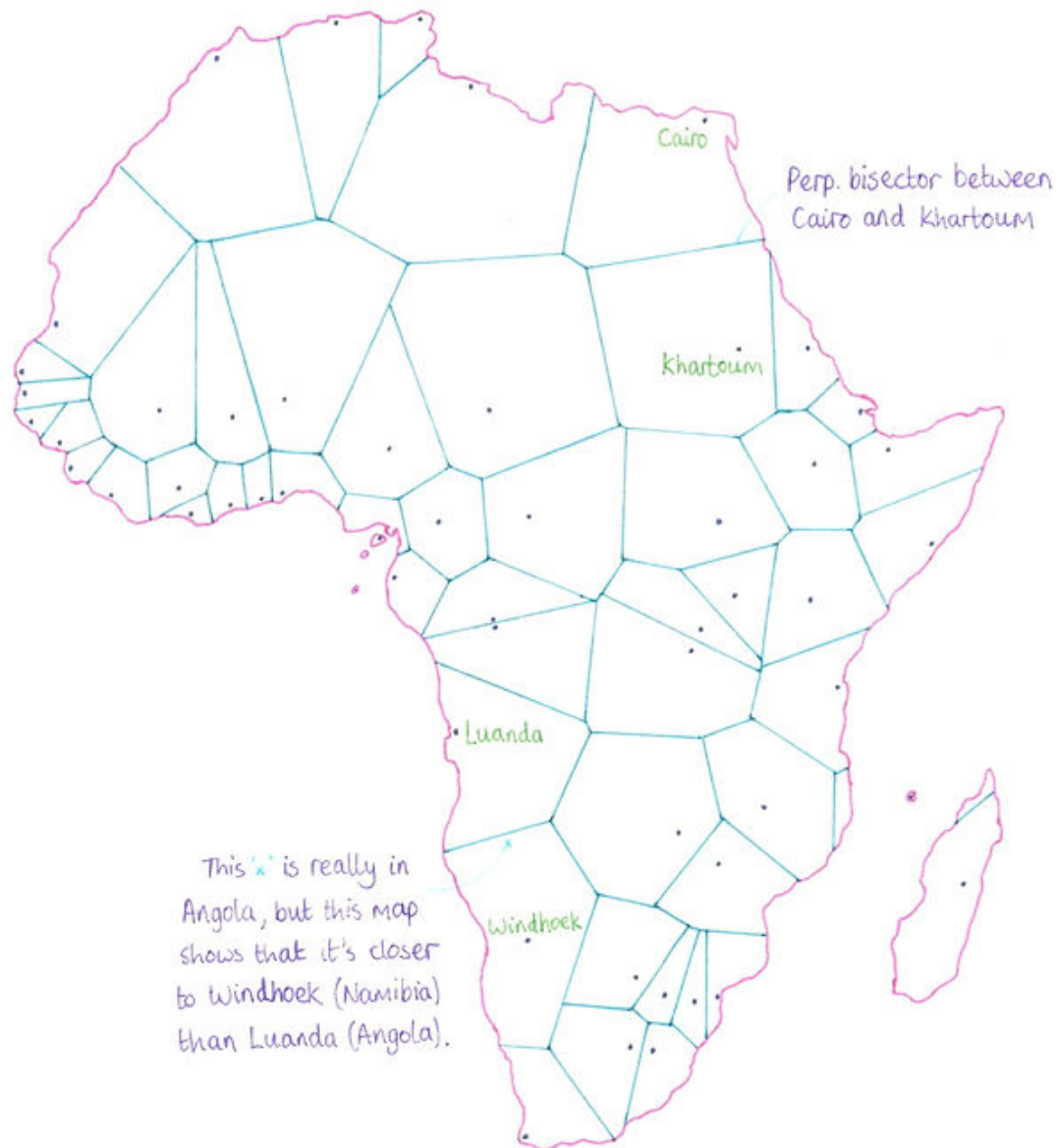
Finally, you will learn about *nearest neighbour interpolation* and the *toxic waste dump problem* on page (3/3) of 3.6.



3.6 continued (2/3)

VORONOI MAP (not required) → One interesting use of Voronoi diagrams is to apply them to maps. This replaces the usual country borders, and changes them to be *cells* that are closest to each capital city.

MAP → Of Africa:



- New 'countries' (cells) now have straight borders, as they are perp. bisectors.
- Countries like D.R.C., Ethiopia and Algeria get much 'smaller'.
- Countries like Eritrea, Uganda and Lesotho get 'larger' – which means that areas of neighbouring countries are actually closer to those three countries' capitals. E.G.: Tigray region of Ethiopia is closer to Asmara than Addis.

3.6 continued (3/3)

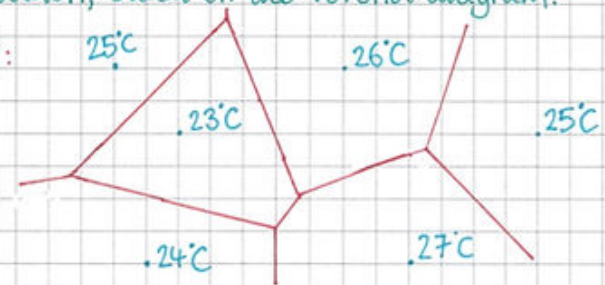
ADDING A SITE $\rightarrow$ This process would need about 10 diagrams to properly show this step-by-step, so I suggest you get someone to walk you through it, or watch a video. But to sum it up briefly:

- Draw a perp. bisector *ray* between the new site & the other site in that cell already
- Draw perp. bisectors with other adjacent sites
- Erase all old line segments that are sitting within the new site's cell. [DONE]

NEAREST NEIGHBOUR INTERPOLATION $\rightarrow$ This may be used by weather apps to give you their best estimate of your location's temperature or rainfall. This method simply uses the nearest station, based on the Voronoi diagram.

E.G. 2 $\rightarrow$ Estimate the temperature at P:

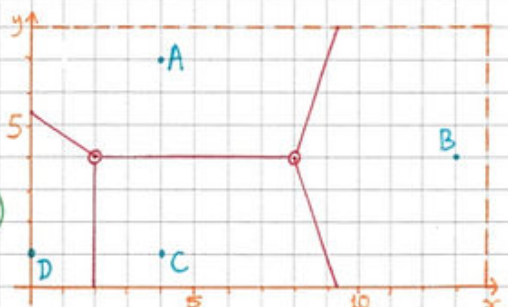
$\rightarrow$ The temperature reading from the station in the same cell as P says 26°C, so that is our best estimate for P.



TOXIC WASTE DUMP PROBLEM $\rightarrow$ This refers to the problem of: which point is furthest away from all sites? Strangely named due to a potential company trying to 'dump toxic waste' as far away from any house or town as possible. (There are other uses). Given that any point on the perpendicular bisector is equidistant from 2 points, but you should want to move away from the actual midpoint, to get further away from those 2 points, you will eventually reach a vertex. Hence, the centre of the 'LEC' (largest empty circle) will be on a vertex.

E.G. 3 $\rightarrow$ Find the coordinates of the centre of the LEC for the Voronoi in the grid shown:

$\rightarrow$ Either at vertex: (2,4) or (8,4). Test distances to adjacent vertices, then choose. First: (2,4) to A(4,7) $= \sqrt{(7-4)^2 + (4-2)^2} = \sqrt{9+4} = \sqrt{13}$. Dist. from (8,4) to B(13,4) is clearly 5. $\sqrt{13} < 5$, hence (2,4) is our answer.



3.7 - Radians, Arcs & Sectors [AHL]

	SL	HL
AKA	3.4	3.4
AKI		✓

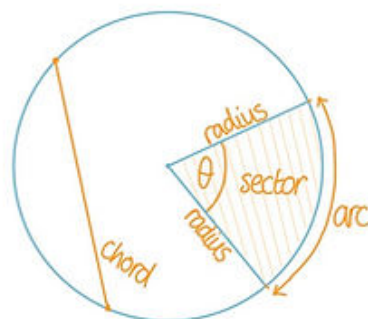
RADIANS → Ever since elementary school, you measured angles in terms of **degrees**, where 360° represents a 'full-turn', and 90° represents a right-angle, etc. (Unless you saw this in the past year or two), now, we learn an alternative to degrees → **radians**. It is just a different way measuring angles, and is a simple linear conversion: $2\pi \text{ radians} = 360^\circ$, $\pi \text{ rad.} = 180^\circ$, $\frac{\pi}{2} \text{ rad.} = 90^\circ$, $\frac{\pi}{6} \text{ rad.} = 30^\circ$, etc... In fact, there is no symbol for radians like there is for degrees. For conversion between the two:

$$\frac{\text{degrees}}{180} \times \pi = \text{radians} \quad \longleftrightarrow \quad \frac{\text{radians}}{\pi} \times 180 = \text{degrees} \quad \rightarrow \text{Neither in F.B.}$$

→ It might seem quite arbitrary, or even pointless, to introduce this new angle measure at this stage. But the first helpful use of radians is shown below. Then, there are further helpful uses in **calculus**, **trig.**, and more.

→ For the mode on your GDC, and for your own thinking, you need to know which one to use. Radians should now be the default, and you'll only go back to degrees if you see the $^\circ$ symbol given in the question.

ARC/SECTOR → You should know how to find the area or circumference of a circle. In the IB, you need to know how find a certain part/section/fraction of the area or circumference. You did this already in **degrees** in 3.4. A part of a circumference is called an **arc** (see diagram). The fraction of the area, in the shape of a pizza slice, is called a **sector** (see diagram).



ARC LENGTH → As a full circle's circumference is found with $C = 2\pi r$, and we are considering $\frac{\theta}{360}$ or $\frac{\theta}{2\pi}$ of the circumference, then the arc length is $2\pi r \times \frac{\theta}{2\pi}$ →

$$\text{Arc Length} = \theta r$$

only for RADIANS

SECTOR AREA → If we use exactly the same concept, we would simply do $\frac{\theta}{2\pi} \times [\text{area of a circle}] \rightarrow \frac{\theta}{2\pi} \times \pi r^2$, which also cancels nicely to get the formula →

$$\text{Sector Area} = \frac{1}{2} \theta r^2$$

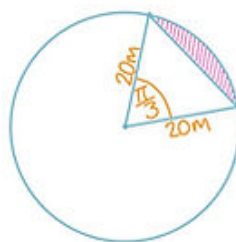
→ In F.B.

E.G. 1 → Find the arc length & sector area shown in the diagram below:



$$\begin{aligned} \text{Arc length (L)} &= \theta r \\ L &= \frac{2\pi}{3} \times 8 \approx \underline{16.8 \text{ cm}} \\ \text{Sector Area (A)} &= \frac{1}{2} \theta r^2 \\ A &= \frac{1}{2} \left(\frac{2\pi}{3} \right) \cdot 8^2 \approx \underline{67.0 \text{ cm}^2} \end{aligned}$$

E.G. 2 → Find the shaded area:



$$\begin{aligned} \text{Shaded area} &= \text{Sector} - \text{Triangle} \\ \text{Sector} &: \frac{1}{2} \left(\frac{\pi}{3} \right) (20)^2 = 209.4395... \\ \text{Triangle} &: \frac{1}{2} (20)(20) \sin \frac{\pi}{3} = 173.2051... \\ 209.4395 - 173.2051 &\approx \underline{36.2 \text{ m}^2} \end{aligned}$$

3.8 - Unit Circle & Identities [AHL]

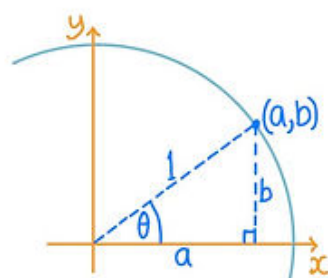
	SL	HL
A&A	3.4	3.4
A&I		✓

UNIT CIRCLE → This is a crucial tool, fundamental to trigonometry. It is a circle with its centre at the origin, with a radius of 1. We can analyse the right-angled triangle created by the point (a, b) :

Using SOH CAH TOA:

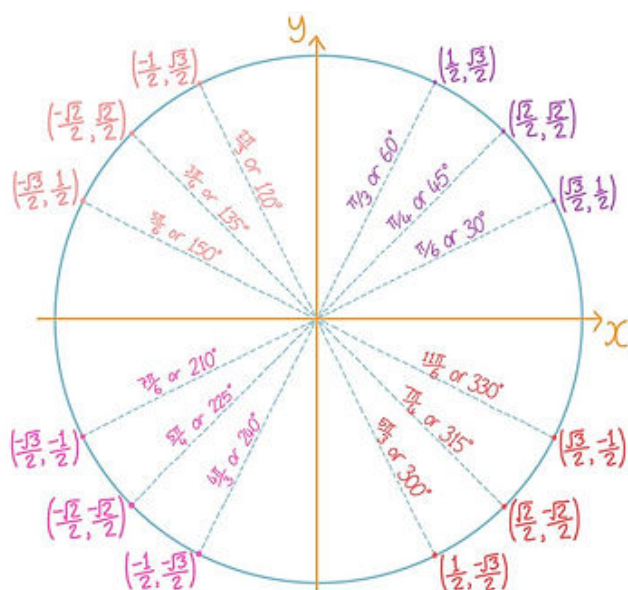
$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{1} = b$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{1} = a$$



So we get the result that $(a, b) = (\cos \theta, \sin \theta)$

→ The implication of this is that if you pick any point on the unit circle, and let the line connecting to it from the origin be at angle θ , then the x -coordinate is $\cos \theta$, and the y -coordinate is $\sin \theta$.



SPECIFIC VALUES → This tells you $\cos 60^\circ = \frac{1}{2}$, or that $\sin(\frac{\pi}{6}) = \frac{1}{2}$. You may have had to learn these if you've done much trig before IB, and you do have to memorise these in A&A. To be clear, in A&I, you **do not** need to.

→ Due to the periodic nature of sine & cosine (and the unit circle), the values shown here repeat every 2π radians (360°).

E.G. 1 → $\sin(\frac{13\pi}{6}) \rightarrow \sin(\frac{13\pi}{6}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$

TAN → Looking back at the triangle at the top, we can also use our knowledge of SOHCAHTOA to say that $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$, but given that $b = \sin \theta$ & $a = \cos \theta$, then:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \rightarrow \text{In F.B.}$$

This will allow us to find all the exact values of $\tan \theta$ for all the angles given in the unit circle above, but will also help significantly when proving identities or solving trig equations.

AMBIGUOUS CASE: SINE RULE → You can see above that the values for sine from 0° to 90° are repeated and reversed for the angles 90° to 180° . This means, when using the sine rule, you could get **two valid answers**.
→ So either accept both, or choose one based on stated restrictions or based on the context of the problem.

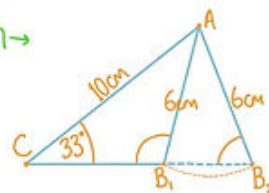
E.G. 2 → If $\angle ACB = 33^\circ$, $AC = 10\text{cm}$, and $AB = 6\text{cm}$, find $\angle ABC$, if it is obtuse:

→ Two possibilities shown →

$$\frac{\sin ABC}{10} = \frac{\sin 33^\circ}{6}$$

$$ABC = \sin^{-1}(0.9077\dots)$$

$$\rightarrow AB_1C = 65.2^\circ, AB_2C = 180^\circ - 65.2^\circ \approx \underline{115^\circ}$$



3.8 continued

NO. OF SOLUTIONS → If you look at a sine curve, you may notice that it passes the vertical line test, but fails the horizontal line test. This means that $y = \sin x$ is a function, but its inverse isn't. More relevantly, it tells us that when you sine of an angle you get one result, but when you do $\sin^{-1}$ of a value (to find a missing angle), you get multiple results, in fact you get infinite answers/angles.

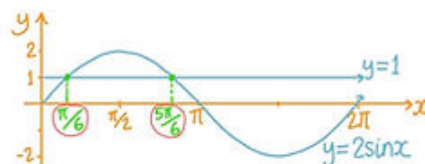
RESTRICTED DOMAIN → You could, in theory, describe your infinite answers. For example, as $\tan x$ repeats every π radians. So you could have: $\tan x = 1 \rightarrow x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}$. It is far more likely that they will give you an interval in which your angle, or angles, must fall in between.

TIP → Consider many angles for all your steps, then discount unwanted ones at the very final step.

GRAPHING → While graphing sometimes a little extra time, it can really clarify which results to select.

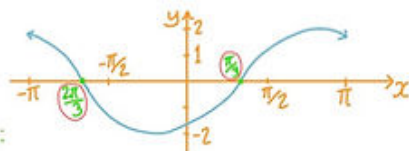
E.G.3 → Solve $2\sin x = 1, 0 \leq x \leq 2\pi$: → BY GRAPHING:

→ ALGEBRAICALLY: $2\sin x = 1 \rightarrow \sin x = \frac{1}{2}$
 $\rightarrow x = \sin^{-1}(\frac{1}{2}) \rightarrow \cancel{\frac{\pi}{6}}, \frac{\pi}{6}, \frac{5\pi}{6}, \cancel{\frac{7\pi}{6}}$
 Graph $y = 2\sin x$ & $y = 1$ in correct domain:



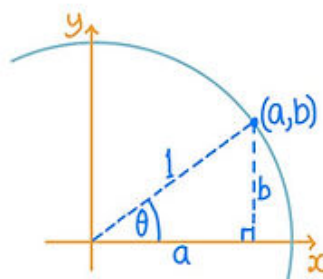
E.G.4 → Solve $\sin x = \sqrt{3}\cos x, -\pi \leq x \leq \pi$: → BY GRAPHING:

→ $\sin x - \sqrt{3}\cos x = 0 \rightarrow \frac{\sin x}{\cos x} - \frac{\sqrt{3}\cos x}{\cos x} = 0$
 $\rightarrow \tan x - \sqrt{3} = 0 \rightarrow \tan x = \sqrt{3} \rightarrow \frac{-2\pi}{3}, \frac{\pi}{3}, \cancel{\frac{4\pi}{3}}$
 This time, we can graph $y = \sin x - \sqrt{3}\cos x$ & find roots:



PYTHAGOREAN IDENTITY → Going back to the right-angled triangle in the unit circle again, and the only bit of geometry left for us to do the Pythagorean theorem on it. In this case, $a^2 + b^2 = 1^2$, but recall that $a = \cos \theta$ & $b = \sin \theta$, so we get:

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \text{In F.B.}$$



E.G.5 → If $\sin \theta = \frac{3}{4}$, find $\cos \theta$:

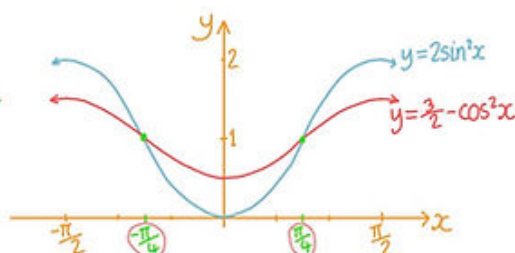
→ As $\sin^2 \theta + \cos^2 \theta = 1 \rightarrow (\frac{3}{4})^2 + \cos^2 \theta = 1 \rightarrow \cos^2 \theta = 1 - \frac{9}{16} \rightarrow \cos \theta = \pm \frac{\sqrt{7}}{4}$

E.G.6 → Solve $2\sin^2 x = \frac{3}{2} - \cos^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$:

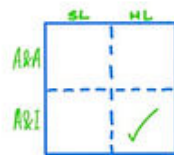
→ Isolate $\sin^2 x + \cos^2 x$: $2\sin^2 x = \frac{3}{2} - \cos^2 x \rightarrow 2\sin^2 x + \cos^2 x = \frac{3}{2} \rightarrow \sin^2 x + (\sin^2 x + \cos^2 x) = \frac{3}{2}$
 $\xrightarrow{\text{sub. in 1}} \sin^2 x + (1) = \frac{3}{2} \rightarrow \sin^2 x = \frac{1}{2} \rightarrow \sin x = \pm \frac{\sqrt{2}}{2} \rightarrow x = \frac{\pi}{4}, \frac{\pi}{6}, \cancel{\frac{5\pi}{6}}, \dots$

OR BY GRAPHING:

Graph $y = 2\sin^2 x$ & $y = \frac{3}{2} - \cos^2 x$ and find intersections:



3.9 - Matrix Transformations [AHL]



MATRIX MULTIPLICATION → If we take a point on the plane, (x, y) , and write it as a position vector,

$\begin{pmatrix} x \\ y \end{pmatrix}$. Then, if you recall from [1.4], multiplying this by a 2×2 matrix, would give us another 2×1 position vector.

E.G. 1 → Multiply $\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \cdot 5 + 1 \cdot 2 \\ 3 \cdot 5 - 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$ → So, in a way, we have moved $(5, 2)$ to $(12, 7)$ through matrix mult.

→ However, this isn't a helpful type of transformation. What would be helpful would be moving it to $(-5, 2)$, $(-5, -2)$, $(10, 4)$, etc., as these represent things like reflection, rotation, and enlargement. But with which matrices?

ENLARGEMENT → To 'enlarge' a point (stretch it from the origin), I want to pick a matrix that just multiplies the x -coord. by k , and multiplies y by k too, and nothing else. We can just use $kI = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$. For example, to stretch $(5, 2)$, S.F. 2, then we multiply $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 0 \cdot 2 \\ 0 \cdot 5 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$. Now we can find matrices for many other transformations:

TRANSFORMATIONS → (Pre-)multiply position vectors by these matrices to carry out the listed transformations:

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \text{horizontal stretch, scale factor } k$$

$$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \rightarrow \text{vertical stretch, scale factor } k$$

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \rightarrow \text{enlargement, S.F. } k, \text{ centre } (0, 0)$$

→ ALL in FB. →

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \rightarrow \text{reflection in the line } y = (\tan \theta)x$$

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \rightarrow \text{anticlockwise rot. by } \theta \text{ about } (0, 0)$$

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \rightarrow \text{clockwise rotation by } \theta \text{ about } (0, 0)$$

→ $\theta > 0$

E.G. 2 → Reflect $A(2, 3)$ in the line $y = x$ using matrices:

→ Our only reflection is in the line $y = (\tan \theta)x$. So we find the θ that gives $\tan \theta = 1$, which is $\theta = \frac{\pi}{4}$. So we use $\frac{\pi}{4}$ in the matrix: $\begin{pmatrix} \cos(2 \cdot \frac{\pi}{4}) & \sin(2 \cdot \frac{\pi}{4}) \\ \sin(2 \cdot \frac{\pi}{4}) & -\cos(2 \cdot \frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \therefore A'$ is at $(3, 2)$

COMPOSITE TRANSFORMATIONS → Given the nature of matrices, when points are multiplied by AB , then we do the B transformation, followed by A . Note: We can also add, to do translations. An affine transf. is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$

SHAPES → We can transform whole shapes by simply putting the points next to each other in a $2 \times n$ matrix & multiply.

E.G. 3 → Take the triangle ABC , where $A(1, 2)$, $B(6, 3)$ & $C(3, 5)$. Stretch vertically S.F. 3 then rotate 90° c.w. about $(0, 0)$:

→ ABC has a matrix $\begin{pmatrix} 1 & 6 & 3 \\ 2 & 3 & 5 \end{pmatrix}$. We use $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, then $\begin{pmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We do $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 6 & 3 \\ 2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 15 \\ -1 & -6 & -3 \end{pmatrix}$ is $A'B'C'$

AREA → If an object is transformed by matrix A , then: Area of image = $|\det A| \times$ Area of original

E.G. 4 → **IB** → Pentagon P , with area of 7 cm^2 , is transformed by matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$. Find the area of the image of P :

→ Area of $P' = |\det A| \cdot 7 = (3 \cdot 2 - 0) \cdot 7 = 42$

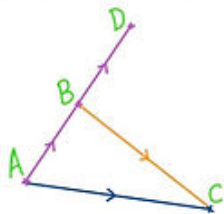
IB → Matrix $B = \frac{1}{2} \begin{pmatrix} 3\sqrt{3} & 3 \\ -2 & 2\sqrt{3} \end{pmatrix}$ is the combined effect of transforming by matrix X , then by A . Find the matrix X .

→ $B = AX \rightarrow X = A^{-1}B \rightarrow X = \begin{pmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{pmatrix}$ by GDC, or $= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

3.10 - Vector Basics [AHL]

	SL	HL
AA&A		3.12
AA&I		✓

VECTORS → This was its own standalone topic in the old SL & HL IB courses. Now it is just part of topic 3 in AAHL & AIHL. **Vectors** are a way of representing quantities with *size & direction*.



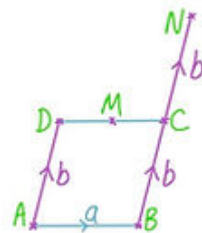
NOTATION → All the information needed to get from A to B can be represented by the displacement vector: $\vec{AB}$

COMBINED → The blue line can be written as: $\vec{AB} + \vec{BC}$, as that gets you from A to C. i.e.: $\vec{AB} + \vec{BC} = \vec{AC}$.

EQUAL VECTORS → As $\vec{AB}$ & $\vec{BD}$ are the same length & direction, this means we can say: $\vec{AB} = \vec{BD}$

→ And as $\vec{AD}$ is twice the length of $\vec{AB}$, we can say: $\vec{AD} = 2 \cdot \vec{AB}$

→ As $\vec{BA}$ is the same length as $\vec{AB}$, but in the *opposite* direction, then: $\vec{BA} = -\vec{AB}$



OTHER NOTATION → Instead of writing $\vec{AB}$ or $\vec{DC}$, we can just use the letter *a*. i.e.: $a = \vec{AB}$

COMBINED → This makes the combinations simpler to write:

$$\vec{AC} = a + b$$

$$\vec{AM} = b + \frac{1}{2}a$$

$$\vec{AN} = a + 2b$$

$$\vec{NM} = -b - \frac{1}{2}a$$



UNIT VECTOR NOTATION → We can use *i*, *j* & *k* to represent a vector. In 2D space, *i* represents moving one unit to the 'right' (or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$), and *j* is 1 unit 'up' (or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$). For example, looking at the diagram to the left, we get: $\vec{PQ} = 4\mathbf{i} + 2\mathbf{j}$



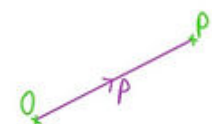
3D → In three dimensions, we view *i* & *j* as 'on the ground', then *k* is in the 'z'-direction, which lifts up off the ground.

COMMON NOTATION → **Column vectors** are possibly the most commonly used form of vectors. From the above example, we get the following column vectors:

$$\vec{PQ} = 4\mathbf{i} + 2\mathbf{j} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\vec{QR} = \mathbf{i} - 3\mathbf{j} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

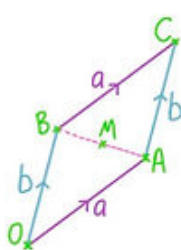
3D → We can do the following: If $\vec{XY} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$



NEGATIVE VECTOR → If the vector taking you from the origin O to P, is given by *p*, then the vector from P to O is given by $-p$.

POSITION VECTOR → Is the vector from the origin to a point. If we have $P(4, 2)$, then $\vec{OP} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

3.10 continued



DISPLACEMENT VECTOR → How can you get from A to B, in terms of a & b ? Well, you could go 'backwards' down a , then go along b in the normal direction. What this really means is $-a$, then b . So we can say $\vec{AB} = -a + b$ or just $\vec{AB} = b - a$

MIDPOINTS → The position vector of the midpoint of $\vec{AB}$ can be calculated by doing $\vec{OM} = \frac{1}{2}(a+b)$. This can be shown by the fact that $\vec{OC} = a+b$, and M is halfway to C.

E.G.1 → In the diagram above, A is at (4,3), B is at (1,4). Find $\vec{OM}$:

$$\vec{OM} = \frac{1}{2}(a+b) = \frac{1}{2}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 3.5 \end{pmatrix}$$

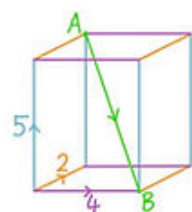
MAGNITUDE → The magnitude of a vector is the length, or displacement of the vector. This is very similar to finding the distance between points, or length of the hypotenuse of the right-angled triangle that can be created by the vector - so we use the **pythagorean thm.**:

→ **NOTATION** → The magnitude of $\vec{AB}$ is denoted with: $|\vec{AB}|$

→ **CALCULATION** → If you have $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, then $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, or in 2D: $|v| = \sqrt{v_1^2 + v_2^2}$ In F.B.

E.G.2 → Find $|\vec{OA}|$ from the diagram above †

$$|\vec{OA}| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$



E.G.3 → For this 3D diagram (left), find $|\vec{AB}|$:

→ Looking at the 'dimensions', we see that $|\vec{AB}| = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$

$$\text{So, } |\vec{AB}| = \sqrt{4^2 + (-2)^2 + 5^2} = \sqrt{45} = 3\sqrt{5} \approx 6.71$$

UNIT VECTOR → In general, a **unit vector** is one that has a magnitude of 1, whatever the direction. So you may be asked to find the unit vector that is in the direction of some other vector. To do this, we just want to alter the magnitude of a vector - a process called **rescaling**. This can be done by multiplying/dividing your vector by a **scalar**. i.e.: doing $k \cdot \vec{a}$. Making the magnitude 1 is called **normalising**, and we do:

$$\text{Unit vector} = \frac{\vec{a}}{|\vec{a}|}$$

E.G.4 → If $L_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, give a unit vector in the direction of L_2 :

→ This has a magnitude of $\sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$, but we want to change the length to 1, so we divide the vector by $\sqrt{5}$: $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ 0 \\ \frac{\sqrt{5}}{5} \end{pmatrix}$, or rationalised: $\begin{pmatrix} \frac{2\sqrt{5}}{5} \\ 0 \\ \frac{\sqrt{5}}{5} \end{pmatrix}$

3.11 - Vector Equations of Lines [AHL]

	SL	HL
A&A		3.14
A&I		✓

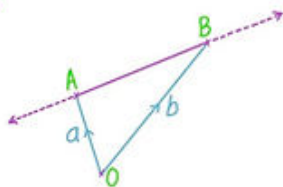
VECTOR EQUATIONS → This is similar to when we have a regular equation of a line in the form $y=mx+c$, with m as the slope & c determining a position (the y-intercept). For vectors, we do:

→ Vector equation of a line:

$$\vec{r} = \underbrace{\begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix}}_{\text{'position vector' } \rightarrow \text{any point on the line}} + \lambda \underbrace{\begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix}}_{\text{'direction vector' } \rightarrow \text{the vector between any 2 points}}$$

E.G. 1 → Find the vector equation of a line passing through A(-1,1,2) & B(3,5,4):

→



→ A position vector can be found using A or B. So: $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ works.

→ The direction vector could be $\vec{AB}$, $\vec{BA}$, or any multiple of these. So let's find $\vec{AB}$:

$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

→ So we could have $\vec{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$ (but there are many correct answers here)

E.G. 2 / 18 2013 → Consider the points: P(-3,-1,2) & Q(5,5,6):

a) Find the vector equation of the line L_1 , passing through P & Q:

$$\rightarrow \text{P.V.} = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} \text{ or } \vec{OP}, \text{ D.V.} = \vec{OQ} - \vec{OP} = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix} \therefore L_1: \vec{r} = \begin{pmatrix} 5 \\ 5 \\ 6 \end{pmatrix} + t \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$$

c) $L_2: \vec{r} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} + s \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$. Find the acute angle between L_1 & L_2 :

→ Take the direction vectors: $\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$ & $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$. Using angle formula from 3.12, we get $\theta = 26.3^\circ$

PARAMETRIC EQUATIONS → This is an alternate way for writing the equation of a line, which involves breaking down the vector equation into its x , y , and z components.

i.e.:

$$\text{If } \vec{r} = \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} + \lambda \begin{pmatrix} x_d \\ y_d \\ z_d \end{pmatrix}, \text{ then } \begin{cases} x = x_p + \lambda x_d \\ y = y_p + \lambda y_d \\ z = z_p + \lambda z_d \end{cases} \text{ 'parametric equations'}$$

E.G. 3 → Change $\vec{r} = \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ into parametric equations:

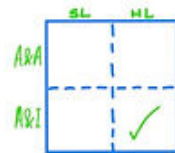
$$\begin{cases} x = 7 + 2t \\ y = -4 \\ z = 1 - 3t \end{cases}$$

E.G. 4 → Find the parametric equations for the line that passes through A(-1,1,3) and is parallel to the vector given by $\vec{v} = 2\vec{i} + 3\vec{j} - \vec{k}$:

$$\rightarrow \text{We get the vector equation: } \vec{r} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \text{ so: } \begin{cases} x = -1 + 2\lambda \\ y = 1 + 3\lambda \\ z = 3 - \lambda \end{cases}$$

→ These are going to be especially important when it comes to finding intersection points, see 3.12.

3.12 - Vector kinematics [AHL]



KINEMATICS → Whilst there are many real-world applications of vector equations, one of the main ones is for movement, or **kinematics**. See 5.13 for more the basic terminology.

→ So the kinematic vector equations will work like this: $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} + t \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ $\mathbf{r}_0$ is still a 'position' vector, but **initial position**
 $\mathbf{v}$ is the **direction/velocity vector**

→ So essentially, the object 'starts at $\mathbf{r}_0$ ', then moves by one $\mathbf{v}$ vector, every sec./hr./etc.

QUESTIONS → This how to find out various bits of info about the movement of this object:

- **Speed**: If moves by one $\mathbf{v}$ every (second), then $|\mathbf{v}|$ is the distance moved each (sec.) ∴ **speed** = $|\mathbf{v}|$
- **Position at a certain time**: Plug in your t -value, and calculate to get your position coordinates.
- **Dist. travelled**: Either find both positions and use distance formula, or do $(t_2 - t_1) \cdot |\mathbf{v}|$, which is **dist. = time × speed**
- **INTERSECTIONS** → This applies to intersections of all vector equations. You'll want to turn the vector eqs. into their **parametric equations**. Then, if they are going to be 'intersecting', then you need to set the x, y , and z -components equal to their counterparts, and solve the system of equations to find the **intersection point**.

E.G. 1 → 2 toy planes have equations $\mathbf{r}_A = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$ and $\mathbf{r}_B = \begin{pmatrix} 39 \\ 44 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix}$. Time (s & t) in seconds after each takeoff. Dist. in m:

a) Find speed of airplane B:

$$\rightarrow |\mathbf{v}| = \sqrt{(-4)^2 + 2^2 + 4^2} = \sqrt{16 + 4 + 16} = \sqrt{36} = 6 \text{ ms}^{-1}$$

b) The toy planes collide at point P. Find P:

$$\begin{array}{l} \mathbf{r}_A = 5 - 4t \\ y_A = 6 + 2t \\ z_A = 4t \end{array} \quad \begin{array}{l} \mathbf{r}_B = 39 - 4s \\ y_B = 44 - 6s \\ z_B = 7s \end{array} \rightarrow \begin{array}{l} 5 - 4t = 39 - 4s \\ 6 + 2t = 44 - 6s \end{array} \xrightarrow{\text{solve}} \begin{array}{l} s = 4 \\ t = 7 \end{array} \rightarrow \mathbf{r}_A = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix} \rightarrow P(-23, 20, 28)$$

→ These might be combined with calculus to ask to minimise distance between objects. **Subtract**, then **minimise dist.**

VARIABLE VELOCITY → As $\mathbf{v}$ above was just a numerical vector, that means those situations had **constant velocity**. Here, we now see vectors describing **variable velocity** - speed changing over time.

STRUCTURE → These may not necessarily follow the usual $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$ structure. You often just see $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ instead. This means you just have a function of t to give you the x & y coordinates separately. You could be asked many of the same types of question as above with these.

CALCULUS → You may know from 5.13 that $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$. The same applies here, but merely separated into x & y components. So we get: If $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, then $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$, and $\mathbf{a} = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}$

E.G. 2 → The position vector of a particle at time t (secs) is given by $\mathbf{r} = 3\cos(3t)\mathbf{i} + 4\sin(3t)\mathbf{j}$, with displacement in metres.

a) Find an expression for the velocity of the particle at time t :

$$\rightarrow \text{Using the chain rule: } \mathbf{v} = -9\sin(3t)\mathbf{i} + 12\cos(3t)\mathbf{j}$$

b) Hence find the speed at $t=3$:

$$\rightarrow |\mathbf{v}| = \sqrt{(-9\sin 9)^2 + (12\cos 9)^2} \approx 11.5 \text{ ms}^{-1}$$

3.13 - Multiplication w/ Vectors [AHL]

	SL	HL
A&A		3.13/16
A&I		✓

→ We know how to multiply numbers, and how to multiply a vector by a scalar, but what about multiplying two vectors? There are a few options for how this could be done, but two ways are relevant:

SCALAR PRODUCT → This is also called the **dot product**. It is one way of 'multiplying' vectors, and it will be useful for assessing angles between vectors, amongst other things. This type of product is calculated as follows:

FORMULA → If $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ & $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then: $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$ → In F.B.

PROPERTIES → $v \cdot w = w \cdot v$
 $u \cdot (v + w) = u \cdot v + u \cdot w$

→ $(kv) \cdot w = k(v \cdot w)$
 $v \cdot v = |v|^2$ → NONE IN F.B.

E.G. 1 → Find $a \cdot b$, if $a = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ & $b = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$:
 $a \cdot b = (2 \times 5) + (1 \times 2) + (2 \times 2) = 16$

E.G. 2 → If $p = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ & $q = \begin{pmatrix} 5 \\ -2 \\ 0.5 \end{pmatrix}$, find $p \cdot q$:
 $p \cdot q = (0 \times 5) + (-1 \times -2) + (2 \times 0.5) = 0 + 2 + 1 = 3$

ANGLE BETWEEN 2 VECTORS → This is found using the following formula, which can be derived using the **cosine rule**:

$$\cos \theta = \frac{a \cdot b}{|a||b|} \rightarrow \text{In F.B.}$$

→ **NOTE** → If you are doing $\cos ABC$, you'll want to specifically use: $\cos ABC = \frac{BA \cdot BC}{|BA||BC|}$

→ If you consider the fact that **parallel** vectors have an angle of 0° between them, and **perpendicular** vectors are at an angle of 90° .

PARALLEL VECTORS → If a & b are **parallel**, then: $a \cdot b = |a||b|$ → NOT IN F.B.

PERPENDICULAR → If a & b are **perpendicular**, then: $a \cdot b = 0$

→ These are both because $\cos(0^\circ) = 1$ and $\cos(90^\circ) = 0$

E.G. 3 → Find angle between $a = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ & $b = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$:
 $\cos \theta = \frac{(5 \times 1) + (1 \times -2) + (2 \times 3)}{\sqrt{5^2 + 1^2 + 2^2} \times \sqrt{1^2 + (-2)^2 + 3^2}} \approx 0.439155 \dots \rightarrow \theta = \cos^{-1}(0.439155 \dots) \approx 1.12 \text{ rad.}$

E.G. 4 → Verify that $u = 10i + 4j - k$ & $w = j + 4k$ are perpendicular:

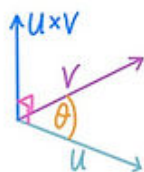
→ Check if $u \cdot w = 0$: $u \cdot w = (10 \times 0) + (4 \times 1) + (-1 \times 4) = 0 + 4 - 4 = 0$

∴ u & w are perpendicular.

→ As mentioned in 3.11, anything involving vector equations and their angles requires you to just focus in on the **direction vectors** of the lines. Otherwise, all rules above apply.

3.13 continued - Vector Products

VECTOR PRODUCT → This is a second type of vector multiplication, also called the **cross product**. Unlike the dot product, the result will be a **vector**, specifically the vector that is **perpendicular** to both u & v (see below):



CALCULATION → To find the cross product, there are formulae involving matrices, but we mainly have:

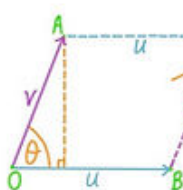
$$u \times v = \begin{pmatrix} u_1 v_3 - u_3 v_1 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \rightarrow \text{IN F.B.}$$

E.G. 1 → Find $u \times v$, if $u = 2i - 3j + k$ and $v = i + 3j - 2k$:

$$\rightarrow u \times v = \begin{pmatrix} (-3)(-2) - (1)(3) \\ (1)(1) - (2)(-2) \\ (2)(3) - (-3)(1) \end{pmatrix} = \begin{pmatrix} 6 - 3 \\ 1 + 4 \\ 6 + 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 9 \end{pmatrix}$$

THEOREM → For angles between $u \times v$, we have $|u \times v| = |u||v|\sin\theta$ → IN F.B.

GEOMETRIC PROPERTIES → The area of a **parallelogram** can be calculated using the **magnitude** of the **cross-product** of adjacent sides:



Area of the parallelogram = base $\times$ height
 $= |u| \cdot |v| \sin\theta$ [using SOHCAHTOA]

$$\text{Area} = |u \times v| \rightarrow \text{IN F.B.}$$

→ Using this rule for areas of a parallelogram, and the fact that you can **bisect** it into two triangles by drawing a line from A to B, we get:

$$\rightarrow \text{Area of triangle (OAB above)} = \frac{u \times v}{2}$$

→ Both parallelograms and triangles are seen in IB exams.

COMPONENTS → Up until this point, we have only been concerned with the direction a vector is pointing, and



how that gets split up into its x & y components. If you consider the box in the diagram being pushed by a force represented by vector a , but it will only move in the direction of b . You can see that it would be more efficient to push in b 's direction, i.e.: some force is lost, and only a 'component' of the force is said to be 'acting' in a 's direction:

$$\text{Component} = \frac{a \cdot b}{|b|} = |a| \cos\theta$$

PERPENDICULAR COMPONENTS → There are also various applications, such as in Physics, where one might want to know the component of vector a that is acting **perpendicular** to vector b .

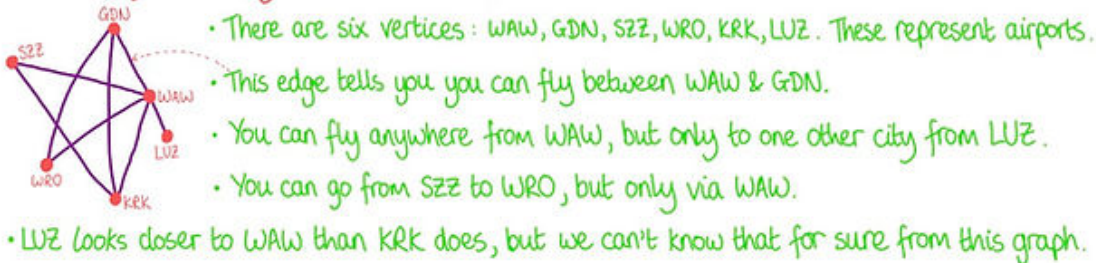
$$\text{Comp.}_{\text{PERP.}} = \frac{|a \times b|}{|b|} = |a| \sin\theta \rightarrow \text{NEITHER IN F.B.}$$

3.14 - Intro to Graph Theory [AHL]

	SL	HL
ARA		
ARI		✓

GRAPHS → These are not 'graphs' in the sense of the word as you've been using it for years, with x & y axes, scatter plots, bar charts, line graphs, or anything like that. A **graph**, in this unit, is a collection of **vertices** (or nodes), and various **edges** that connect them. This can represent almost any link or connection that you might find in real life. For example, roads linking towns, relationships between people, borders between countries, or links between subway stations. The size/length of edges displayed in the diagram have no meaning at all.

E.G. 1 → A graph showing direct domestic flight links in Poland:

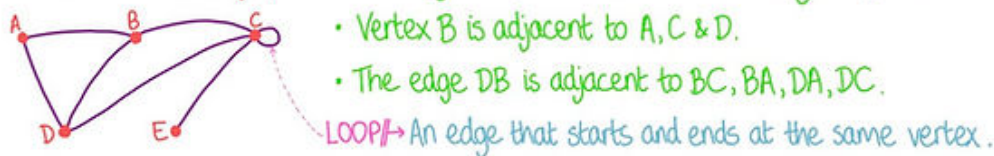


DEGREE → The **degree** of a vertex is the number of edges linked to a vertex.

ADJACENT VERTEX → A vertex is **adjacent** to another if an edge directly links them.

ADJACENT EDGE → An edge is **adjacent** to another if they both connect to the same vertex.

E.G. 2 → An abstract graph: • The degree of vertex B is 3. The degree of E is 1.



SIMPLE GRAPH → Any graph without loops, and with at most one edge going between a pair of vertices.

COMPLETE GRAPH → A graph where every pair of vertices has exactly one edge linking them.

WEIGHTED GRAPHS → The two graphs shown above only show that certain connections exist. However, they do not give any more information than that. However, we can attach a numerical value to each edge. This would not only show the connection exists, but also assigns a magnitude to each edge. This can be:

times, distances, costs, etc. **Time (mins)** →

E.G. 3 → Here is the Polish flight graph shown again with flight times & costs:

→ With these weights, we will solve more complex problems regarding routes between cities.

DIRECTED GRAPHS → As well as weights of edges, there may only be a one-sided connection between vertices. In this case, an arrow on the edge will denote the direction of the edge. For example, if you could only fly from WAW to WRO, but not back, then it would be shown as:



3.14 continued

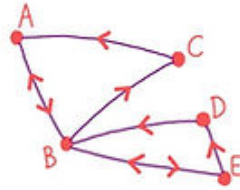
IN-DEGREE → If you are now dealing with a **directed graph**, the concept of the degree of a vertex needs refining. An **in-degree** is the number of edges connected to a vertex that are directed towards it. Naturally, the **out-degree** is simply the number of edges directed away from the vertex.

E.G.4a) Give the in-degree of vertex A:

↳ 2 [C to A, B to A]

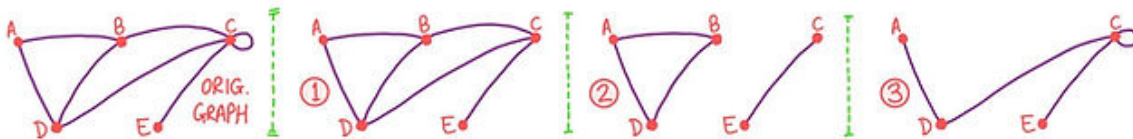
4b) Give the out-degree of vertex B:

↳ 3 [B to A, B to C, B to E]



SUBGRAPH → A **subgraph** of an original graph is a graph comprised of any subset of the vertices and the edges of the original graph.

E.G.5) Here are three subgraphs of the graph from E.G.2:



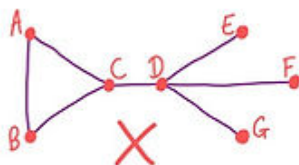
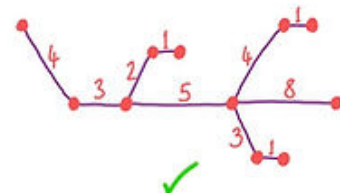
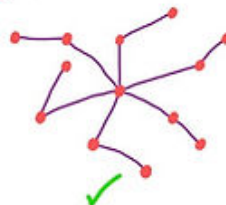
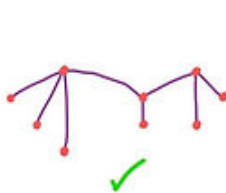
CONNECTED → A graph is said to be **connected** if a route exists to get from any vertex to any other vertex, using some combination of edges. If there are completely distinct sections to the graph, such as in subgraph ②, then it is **disconnected**.

We will be able to define this more formally after 3.16.

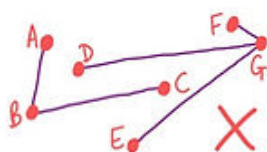
TREES → This is a simple but important concept that will keep cropping up in Graph Theory. A graph will be classified as a **tree** if it is connected, and if it has no 'cycles' (see 3.16)

E.G.6) Here are three examples of graphs that would be classed as trees, and two that wouldn't:

↳



→ that closed triangle ABC is an example of **cycle**, so this is not a tree.



→ you can't get from B to G, so this is not **connected**, so this is not a tree.

3.15 - Adjacency Matrices & Walks [AHL]

	SL	HL
ARR		
ARL		✓

ADJACENCY MATRICES → This will be a way of representing all of the crucial information regarding a graph in the form of a matrix (or table). This might be just the existence of connections between vertices, or the weights of those edges, as well as their directions.

FOR UNWEIGHTED → For clarity, we can start off by looking at an *adjacency table*. Each row and column represents a vertex, and the number in each cell represents how many direct links there are between corresponding vertices. [See E.G.1]

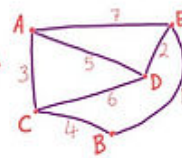
LOOP → You should be able to add up a row to find the degree of that vertex. ∴ A *loop* will be counted as 2 in a matrix.

DIRECTED → The previous matrices were symmetrical, as there are two ends to every edge. When a graph is directed, this is no longer the case. The vertices on the left of a table (row headers) refer to where the edges are coming from. Otherwise, it works in the same way. [See E.G.2]

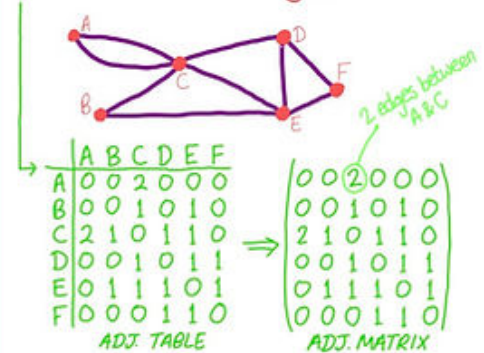
WEIGHTED → So, adjacency matrices can also be used for a completely separate purpose: to show *weights*. Values in each cell simply show the weights of the edges between corresponding vertices.

E.G.3 → Make graph based on the adjacency table:

	A	B	C	D	E
A	0	0	3	5	7
B	0	0	4	0	1
C	3	4	0	6	0
D	5	0	6	0	2
E	7	1	0	2	0



E.G.1 → Make an adjacency table and matrix for the following graph:



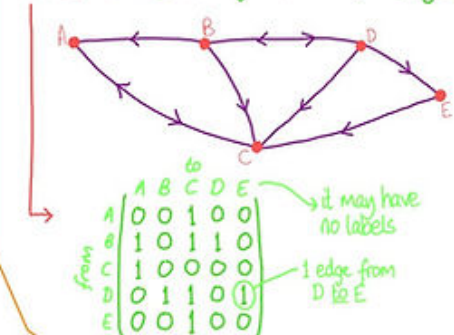
	A	B	C	D	E	F
A	0	0	2	0	0	0
B	0	0	1	0	1	0
C	2	1	0	1	1	0
D	0	0	1	0	1	1
E	0	1	1	0	1	0
F	0	0	0	1	1	0

ADJ. TABLE

0	0	2	0	0	0
0	0	1	0	1	0
2	1	0	1	1	0
0	0	1	0	1	1
0	1	1	0	1	0
0	0	0	1	1	0

ADJ. MATRIX

E.G.2 → Make an adj. matrix of the graph:

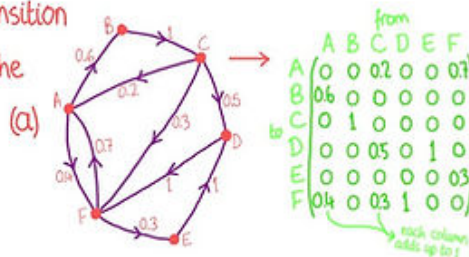


	A	B	C	D	E
A	0	0	1	0	0
B	1	0	1	1	0
C	1	0	0	0	0
D	0	1	1	0	1
E	0	0	1	0	0

from to

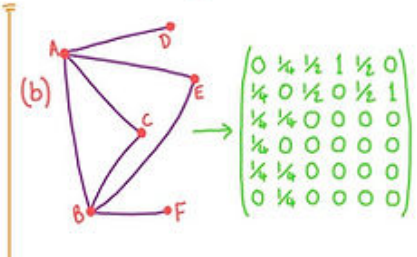
TRANSITION MATRICES → One final concept in this area involves *probability*, as well as content from 4.19 about transition matrices. Here, they are used to show probabilities for which vertices may be travelled to next, after reaching a certain vertex, if the choice is *random*. These probabilities will either be given explicitly as weights, or if not, it will be assumed that each available edge is equally likely.

E.G.4 → Make a transition matrix for each of the following graphs:



	A	B	C	D	E	F
A	0	0	0.2	0	0	0.7
B	0.6	0	0	0	0	0
C	0	1	0	0	0	0
D	0	0	0.5	0	1	0
E	0	0	0	0	0	0.3
F	0.4	0	0.3	1	0	0

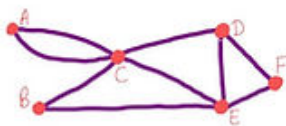
from to



0	1/4	1/2	1	1/2	0
1/4	0	1/2	0	1/2	1
1/4	1/4	0	0	0	0
1/2	0	0	0	0	0
1/4	1/4	0	0	0	0
0	1/4	0	0	0	0

3.15 continued

No. of n -STEP WALKS → If you take the graph below, and have a think about how to get from D to B by going along 2 edges, i.e.: with '2 steps', you could probably just find DCB and DEB by inspection. But what if we needed



$$A = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

to find this using the matrix instead (from E.g. 1)? Well, we would start with D's row (highlighted), and see that we have three 1's, in columns C, E & F. What we want to know next is whether C, E or F link to B. We look in their rows now (highlighted), we can see 1's in the C & E rows.

This happens to be the exact same process for finding the D to B cell in the matrix for $A \times A$ or A^2 . So if the calculation and pattern is used, then the interpretation of the result is also the same. i.e.: a_{db} or $a_{2,2}$ is telling us the number of two-step walks from D to B. So this is 2: DCB and DEB.

$$A^2 = \begin{pmatrix} 4 & 2 & 0 & 2 & 2 & 0 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 7 & 1 & 2 & 2 \\ 2 & 2 & 1 & 3 & 2 & 1 \\ 2 & 1 & 2 & 2 & 4 & 1 \\ 0 & 1 & 2 & 1 & 1 & 2 \end{pmatrix}$$

→ No way to get from A to F in 2 steps.
→ Four ways to go from E, back to E, in 2 steps:
→ EBE, ECE, EDE, EFE

$$(0 \ 0 \ 1 \ 0 \ 1 \ 1) \times \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0+0+1+0+1+0 = 2$$

→ This will be extended to any power of the adj. matrix, A^n , giving the number of n -step walks.

E.G. 5a → E-Z-Bus has six stations in Germany, and the direct links shown. Rafa is visiting Dresden for a month and has a voucher for 5 bus journeys. Can he leave Dresden, take 5 buses, and get back to Dresden? How many different ways?



Firstly, we need to make the adj. matrix for this graph:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Then, we use our GDC to find A^5 :

$$A^5 = \begin{pmatrix} 56 & 91 & 83 & 47 & 43 & 43 \\ 91 & 108 & 99 & 65 & 74 & 74 \\ 83 & 99 & 82 & 52 & 65 & 65 \\ 47 & 65 & 52 & 30 & 35 & 35 \\ 43 & 74 & 65 & 35 & 32 & 32 \\ 43 & 74 & 65 & 35 & 32 & 32 \end{pmatrix}$$

We get 30, meaning there are 30 five-step walks from Dresden, back to Dresden. (Note: these might go through Dresden too)

5b → How many ways are there of getting from Augsburg to Frankfurt in at most 3 bus trips? → If A gives the 1-step walks, A^2 shows 2-step & A^3 shows 3-step, then $A+A^2+A^3$ shows the no. of walks of at most 3 steps:

$$A^2 = \begin{pmatrix} 3 & 2 & 1 & 1 & 2 & 2 \\ 2 & 5 & 3 & 1 & 1 & 1 \\ 1 & 3 & 4 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 & 2 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 4 & 9 & 9 & 5 & 3 & 3 \\ 9 & 8 & 9 & 7 & 8 & 8 \\ 9 & 9 & 6 & 4 & 7 & 7 \\ 5 & 7 & 4 & 2 & 3 & 3 \\ 3 & 8 & 7 & 3 & 2 & 2 \\ 3 & 8 & 7 & 3 & 2 & 2 \end{pmatrix}$$

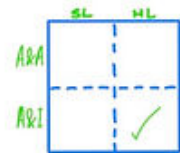
$$A + A^2 + A^3 = \begin{pmatrix} 7 & 12 & 11 & 7 & 5 & 5 \\ 12 & 13 & 13 & 9 & 10 & 10 \\ 11 & 13 & 10 & 6 & 9 & 9 \\ 7 & 9 & 6 & 4 & 4 & 4 \\ 5 & 10 & 9 & 4 & 4 & 4 \\ 5 & 10 & 9 & 4 & 4 & 4 \end{pmatrix}$$

So, 5 ways

TRANSITION MATRICES & STEADY STATES → Again, this is really covered properly in 4.19, but it uses plenty of the concepts covered above. For example, if you square the transition matrix, you get probabilities for where one may end up after 2-step. The same applies to any exponent. Importantly, with large powers, if the values reach an equilibrium, and all columns are equal, then that is the steady state of the matrix. It will tell you proportions of how often random walks would spend at each vertex, in the long term.

PAGERANK ALGORITHM → Google's algorithm for ranking websites for their search results uses a version of this. You can make a graph showing when site link to others. A steady state matrix shows the effective 'popularity' of websites.

3.16 - Routes & Algorithms [AHL]



ROUTES → [Not an official term], but here we will define all of the different types of 'routes' you can take around the edges & vertices of a graph. It will matter whether you repeat edges/vertices and where you can start/finish.

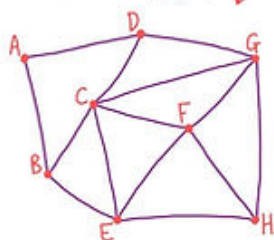
WALK → Literally any route taken around a graph. No restrictions.

TRAIL → A walk, where no edges are repeated. **PATH** → A walk, where no edges or vertices are repeated.

CIRCUIT → A closed trail. i.e.: a trail that starts and ends at the same vertex.

CYCLE → A closed path. i.e.: a path that starts and ends at the same vertex.

E.G.1 → Take the following graph, classify these routes:



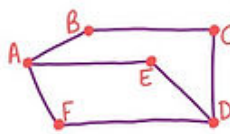
	WALK	TRAIL	PATH	CIRCUIT	CYCLE
ADCFG	✓	✓	✓	✗	✗
CFEHFGC	✓	✓	✗	✓	✗
DCBEFCE	✓	✓	✗	✗	✗
ABEFGDA	✓	✓	✓	✓	✓
GCDCFC	✓	✗	✗	✗	✗

EULERIAN → An **Eulerian circuit** is a circuit that uses every edge. As it is a circuit, remember this means that it can't repeat any edges, and must start and end at the same vertex.

→ An **Eulerian Trail** is simply a trail that uses every edge exactly once.

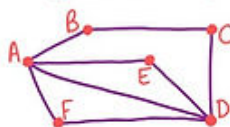
→ A graph itself is **Eulerian** if a Eulerian circuit exists. A graph is **semi-Eulerian** if an Eulerian trail exists.

E.G.2a → Give an Eulerian trail for the following graph:



→ ABCDEAFD is an Eulerian trail, using every edge, without repeating any.

2b → Give an Eulerian circuit for the following graph:



→ ADFAEDCBA is an Eulerian circuit.

TESTING → There is a way of being able to tell if a graph is Eulerian, or semi. If you think about an Eul. circuit, the route must arrive at, and leave from every vertex, and do it on different edges. It may do this multiple **pairs** of times at a certain vertex, but it can only happen in pairs. Therefore:

If every vertex has even degree, then the graph is Eulerian

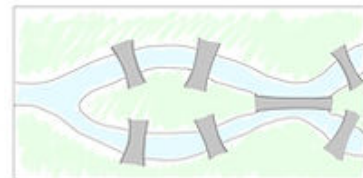
→ That reasoning applies to Eulerian trails, except for the first vertex and last vertex, which would have **odd** degree.

If exactly two vertices have odd degree, then it is semi-Eulerian

E.G.3 → Drawing this graph in one single continuous line would be an Eulerian trail:



E.G.4 → The Bridges of Königsberg is a famous example of a non-Eulerian graph: (all odd degree)



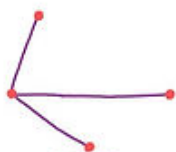
3.16 continued - Minimum Spanning Trees

TREE → Recall, a tree is a connected graph with no cycles.

SPANNING TREE → A spanning tree can only be defined in relation to an original tree. It is a subgraph of the original graph, that is a tree, and uses ('spans') all vertices. The one with the lowest total weight is the **minimum spanning tree**.



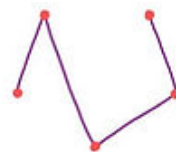
ORIGINAL GRAPH



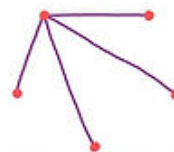
A TREE THAT DOESN'T 'SPAN'



A SPANNING SUBGRAPH, BUT NOT A TREE



SPANNING TREE ✓



SPANNING TREE ✓

ALGORITHMS → In this section, we will see two **algorithms** for finding the minimum spanning tree of a graph. To clarify, algorithms are specific sets of instructions used to achieve a goal or solve a problem. There aren't really any choices or creativity involved. This is why we get computers to help us run algorithms to solve complex problems within Graph theory frequently. For finding the **minimum spanning tree**, we have **Prim's** and **Kruskal's** algorithms:

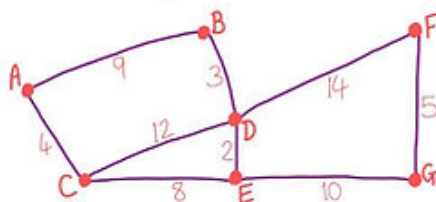
PRIM'S → A vertex-centric approach:

- ① → Start with any vertex (although the IB will tell you which vertex)
- ② → Link it to the nearest vertex (if equal, pick one)
- ③ → Link to the nearest vertex that is connected to either of the first two vertices.
- ④ → Continue joining connected vertices to the nearest unconnected vertex until all verts. are connected.

KRUSKAL'S → An edge-centric approach:

- ① → Pick the shortest edge in the graph (if equal, pick one)
- ② → Choose the next shortest edge
- ③ → Choose the shortest edge that does not complete a cycle.
- ④ → Repeat step 3 until $n-1$ vertices have been selected, given there are n vertices in the graph.

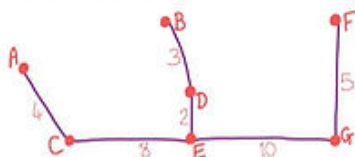
E.G. 5 → Find the minimum spanning tree and its weight for the following graph, using both of the given algorithms:



PRIM'S → Let's start with A:

- Link to C (as $AC < AB$, $4 < 9$)
- Choosing between AB, CD, CE. Pick CE
- Next: ED, then: DB, then: EG, then: GF

M.S.T. →



KRUSKAL'S → Must start with DE (2)

- Next shortest edge is DB (3)
- Then: AC (4), then: GF (5), then: CE (8)
- AB is next shortest, but would make a cycle
- So EG (10) is our final [6th] edge to create the MST.

Total Weight of MST = $4 + 8 + 2 + 3 + 10 + 5 = 32$

PRIM'S w/ ADJACENCY MATRICES → It should be noted that Prim's algorithm is workable even when you just provided with the **adjacency matrix**. You can search for the nearest vertex by looking down corresponding columns, then crossing rows of vertices that have already been chosen. Kruskal's from an A.M. is much trickier, so isn't required.

3.16 continued - Chinese Postman Problem

CHINESE POSTMAN PROBLEM → This is the problem of how far you would have to travel to leave a vertex, traverse every edge at least once, and get back to where you started. As the name suggests, this is like trying to figure out the shortest route to go past every house along every road in their neighbourhood. Obviously, this means they'll want to minimise the number of roads they drive down having already delivered to them. The same applies to repeating edges on graphs.

- ① → The ideal scenario is if you're able to just visit every edge exactly once. This would simply be an **Eulerian Circuit**. So if one exists, then that is the solution to the problem.
- ② → If it is just **semi-Eulerian**, then you can only complete an Eulerian Trail. This means you would have traversed every edge already, but won't be back at the start. You then just find the shortest path back to the start & add that on.
- ③ → If >2 vertices have odd degree, you will need to repeat paths between various pairs of these odd vertices. So, find the combination of paths between pairs that has the shortest total weight. Those edges will be the ones being repeated, so add their weights to the original total weight.

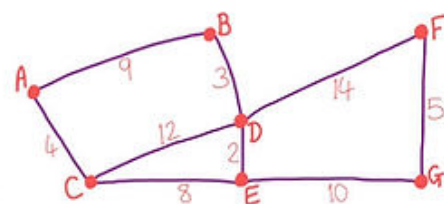
NOTE → For relatively complicated graphs, you will probably just be asked for the weight, rather than the actual route.

NOTE 2 → When you repeat these edges, you can view it as being equivalent to adding extra edges between those vertices. Doing that, would essentially make every vertex have an even degree, which gives us an Eul. Circuit (effectively).

E.G. 6 → Find the shortest way of: starting at C, traversing every edge at least once, and returning back to C. [Graph from E.G. 5]:

↳ We see that C and E are the two vertices with odd degree. So, we must repeat the shortest path between C & E. We confirm that

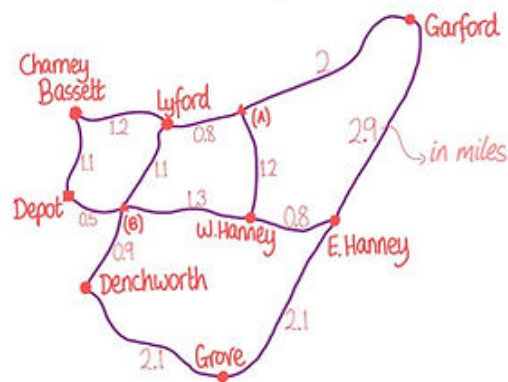
it is the direct edge between the two (weight=8). Route: CABDECDGEC. Total weight = $67 + 8 = 75$



E.G. 7 → The West Oxfordshire county rubbish collection service have reshuffled, and their 3rd district have got a new assignment and map. Naturally, they must collect from every house along every road, between these villages. Suggest the optimal route that the collectors should take to cover their district each week, starting and finishing at the 'depot'.

↳ First, count the odd degree vertices -

These are: Lyford, Jct(A), W.Hanney, E.Hanney. [We'll call these L, A, W & E]. We will need to repeat paths between two pairs of vertices. These could be: L to W + A to E, L to E + A to W, or L to A + W to E. By inspection, we find the sets of shortest paths are $2.0 + 2.0 = 4.0$, $2.8 + 1.2 = 4.0$, and $0.8 + 0.8 = 1.6$ respectively. Clearly, the final option is the lowest total weight, so those are the roads that we'll be willing to drive along for the second time. Every other edge will be used once in our solution: DCLAGaEGrDBWEWALBD would work. The weight is just the graph's total weight + the repeated 1.6. So, we must travel $(1.1 + 1.2 + 0.8 + \dots + 0.5) + 1.6 = 18.0 + 1.6 = 19.6$ miles for the whole route



3.16 continued – Travelling Salesman Problem

➔ The final type of problem in this unit is the **Travelling Salesman Problem**. This is similar to the Chinese Postman, except we are trying to devise the quickest route to visit every **vertex** at least once and return to the start.

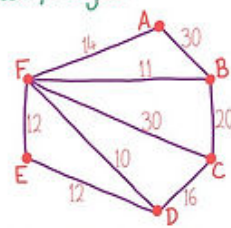
SOLUTION ➔ Note that we will not see a quick algorithm for finding this optimal solution, as it doesn't exist. Instead, we have algorithms for finding **upper & lower bounds** for the total weight of the route.

TYPES ➔ In the T.S.P., we call it the **classical T.S.P.** if you are solving it for a **complete graph** (K_n) – one with edges between every single pair of vertices, such as  or . For classical, the solution is the shortest **Hamiltonian cycle**.
If it is any other type of graph, it is the **practical T.S.P.**, and you may visit vertices more than once if needed.

HAMILTONIAN ➔ A **Hamiltonian Cycle** is a cycle that visits every vertex exactly once. If a graph has a Ham. cycle, then the graph is **Hamiltonian**. A path that visits every vertex exactly once is a **Hamiltonian Path** & a graph that has one is **Semi-Hamiltonian**. Unlike Eulerian, there is no easy test to find these, so we have to use inspection.

TABLE OF LEAST WEIGHTS ➔ We have algorithms for the classical T.S.P., so we can find a way of effectively turning any graph into a complete one. When there isn't a direct edge between two vertices, we consider the **shortest path** between each pair of vertices. These paths almost take the place of edges.

E.G.8 ➔ The graph shows a map of six towns, where a bank operates, and the roads between them. Weights are driving times (mins). Complete the table of least weights:



	A	B	C	D	E	F
A	-		40	24		14
B		-	20	21	23	11
C	40	20	-		28	26
D	24	21		-	12	10
E		23	28	12	-	12
F	14	11	26	10	12	-

➔ CDF is less than the direct edge CF (30)

➔ A to B: via F is best, $11 + 14 = 25$. A to E, via F, $14 + 12 = 26$. C to D is clearly 16.

NEAREST NEIGHBOUR ALG. ➔ For upper bounds

- ① ➔ Complete the table of least weights if needed.
- ② ➔ Start at some vertex (the TB should tell you which)
- ③ ➔ Always move to the next nearest vertex that doesn't complete a cycle.
- ④ ➔ When all vertices have been selected, take the shortest path back to the start. This is an U.B.

DELETED VERTEX ALG. ➔ For lower bounds

- ① ➔ Choose a vertex, 'delete' it, as well as all its edges.
- ② ➔ Find the minimum spanning tree of the remaining graph.
- ③ ➔ The lower bound is equal to the weight of the MST plus the two lowest deleted edge weights.
- ④ ➔ Starting again with another vertex could raise the L.B.

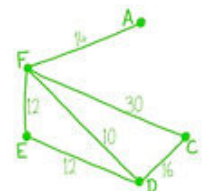
E.G.9 ➔ The bank manager wants to inspect the branches in the five other towns in the region, then return back to the headquarters in town B. Find a lower & upper bound for their inspection route.

NEAREST NEIGHBOUR ALG. ➔

- Start at B, go to F (11), go to D (10), then E (12).
- Next would be A, via F, which we know from the ToLW is 26.
- Then C, via F & C (40).
- Then we go back to B (20).
- So the upper bound is 119 mins, 1hr 59m.

DELETED VERTEX ALG. ➔

- Delete B, and edges AB, FB, CB:
- Find MST with Kruskal's:
 - ➔ Pick FD (10), FE (12), FA (14), DC (16). Total = 52
- Shortest two deleted edges: 11 & 20.
- Lower bound = $52 + 11 + 20 = 83$ mins, 1hr 23m.



➔ To clarify those findings: an optimal route *does* exist, and it's somewhere between 1hr 23m and 1hr 59m.