TOPIC 2 FUNCTIONS

	SL	HL
A&A	✓	✓
A&I	✓	✓

2.1 - Straight Lines

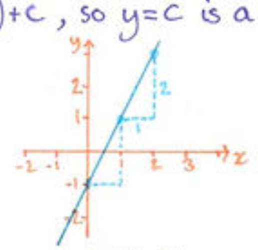
Straight lines may be drawn with a graph of a *linear function*. The form that you have probably seen the most in previous years is $y=mx+c$:

$y=mx+c$ This is the *gradient-intercept form*, and m & c tell us certain properties about the line. As x changes, y changes by 'mx'. $\therefore$ gradient $=\frac{\Delta y}{\Delta x}=\frac{\Delta mx}{\Delta x}=m$. So m is the gradient/slope. If you then set $x=0$ in any function, you can find y-intercepts. In any $y=mx+c$, you get $y=m(0)+c$, so $y=c$ is a y-intercept.

E.G.1 Draw a graph of function: $y=2x-1$

Gradient $=2\left(=\frac{2}{1}\right)$

y-intercept at $y=-1$



$ax+by+d=0$ This is another common form for a straight line, and we can derive it from: $y=mx+c \xrightarrow{-mx-c} -mx+y-c=0$, then if m is a fraction, we can multiply everything by the denominator to get integer coefficients.

E.G.2 Express $y=-\frac{2}{3}x+4$ in the form: $ax+by+d=0$ $(a,b\in\mathbb{Z})$

$\hookrightarrow \frac{2}{3}x+y-4=0 \xrightarrow{\times 3} \underline{2x+3y-12=0}$

$y-y_1=m(x-x_1)$ This is more of a helpful starting point for finding the equation if you have a *point* (x_1,y_1) & a *gradient* (m). As $m=\frac{\Delta y}{\Delta x}=\frac{y-y_1}{x-x_1}$, we get $y-y_1=m(x-x_1)$.

E.G.3 Find equation of a line which is parallel to $y=2x-3$, passing through $(1,3)$:

$\hookrightarrow$ Use $y-y_1=m(x-x_1) \longrightarrow y-3=2(x-1) \rightarrow y-3=2x-2 \rightarrow \underline{y=2x+1}$

PERPENDICULAR If we want to go from a slope of one line to the slope of a *perpendicular* line, then it must change from: $\frac{RISE}{RUN} \rightarrow \frac{-RUN}{RISE}$ or $\boxed{m \rightarrow \frac{-1}{m}}$

E.G.4 Find eq. of a line perpendicular to $y=\frac{5}{2}x-4$, passing through $(10,-3)$:

$\hookrightarrow$ Slope: $\frac{5}{2} \xrightarrow{\frac{-1}{m}} \frac{-1}{5/2}=\frac{-2}{5}$. $y-y_1=m(x-x_1) \rightarrow y+3=\frac{-2}{5}(x-10) \rightarrow y=-\frac{2}{5}x+1$

INTERSECTIONS Two lines' intersection point can be found by 'setting them equal to each other', which is essentially just *substitution*.

E.G.5 Find the intersection point of $y=2x-1$ & $y=-x+8$

$\hookrightarrow$ Solve $-x+8=2x-1$, $9=3x$, $x=3$. Plug-in: $y=2(3)-1 \longrightarrow y=5$ $\therefore \underline{(3,5)}$

2.2 - Function Concepts

	SL	HL
A&A	✓	✓
A&I	✓	✓

// → A function is a series of operations that will transform any x value into one specific (y) value.

NOTATION // → The two main versions of writing a function will be: $y = 4x - 3$ or $f(x) = 4x - 3$. This takes any x value, multiplies it 4, and subtracts 3. For example, $f(2) = 4(2) - 3 = 5$. But, as usual, you may use any letters for variables, such as a 'velocity-time' function like $v(t) = 3t^2 + 2t$.

DOMAIN/RANGE // → An important detail about functions is knowing what set of x values you are able to plug in - *the domain*, and which y values are possible to get out of the function - *the range*. The things that may restrict the domain include that: you can't divide by zero, you 'can't do' the square root of a negative, and you can't do a log of 0 or a negative. The range essentially has an inverse set of restrictions.

E.G. 1 // → Find the domain of $f(x) = \ln(x+1)$:

↳ Must have $x + 1 > 0 \therefore \underline{x > -1}$ alternate notation: $\underline{\{x \in \mathbb{R} \mid x > -1\}}$

E.G. 2 // → Find the range of $g(x) = 2x^2 - 3$:

↳ As $x^2 \geq 0$ for any x, $\underline{g(x) \geq -3}$

INVERSE // → We have seen how a regular function turns x's into y's. The *inverse* of a function turns those y values *back* into their respective x values, reversing the operations. To find these, we can switch x & y, then make y the subject again. Then write it with the notation: $f^{-1}(x) =$ [inverse function]

E.G. 3 // → $f(x) = 2x + 4$, find f^{-1}:

↳ $y = 2x + 4$

switch x & y

$x = 2y + 4$

-4

$x - 4 = 2y$

$\div 2$

$y = \frac{x-4}{2}$

inverse notation

$\underline{f^{-1}(x) = \frac{x-4}{2}}$

E.G. 4 // → $h(x) = \ln(\frac{x}{2})$, find h^{-1}:

↳ $y = \ln(\frac{x}{2})$

switch x & y

$x = \ln(\frac{y}{2})$

raise power of e

$e^x = e^{\ln y/2}$

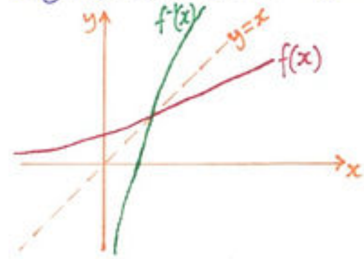
simplify

$e^x = y/2$

$\times 2$

$y = 2e^x \rightarrow \underline{h^{-1}(x) = 2e^x}$

You can graph an inverse function by reflecting the original in the line: $y = x$



2.3 / 2.4 - Key Features of Graphs

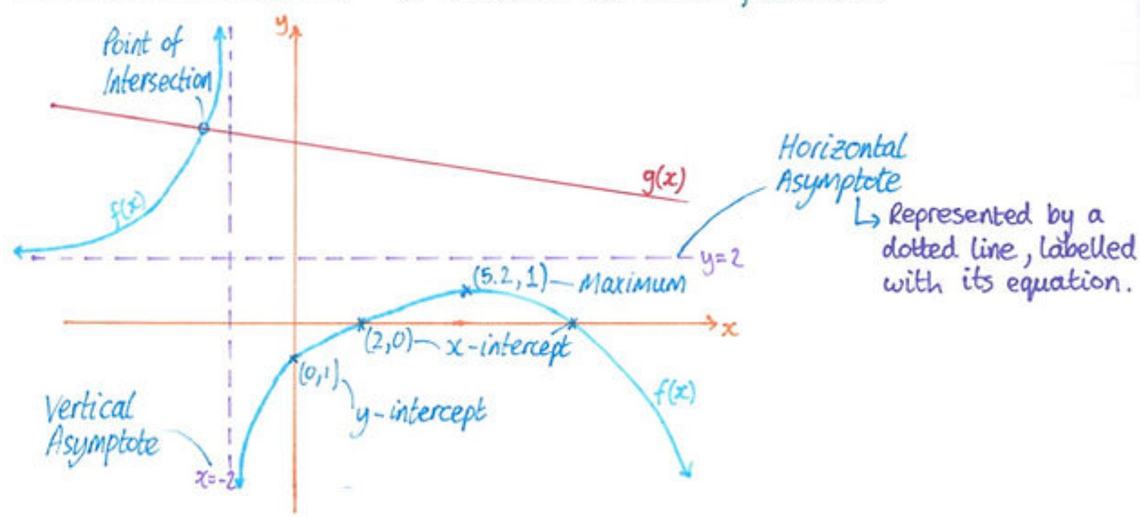
	SL	HL
A&A	✓	✓
A&I	✓	✓

↳ You may well be asked to *draw* or *sketch* the graphs of functions. *Drawing* should be as accurate as possible, including all relevant graph features. *Sketching* should just give a general idea of the shape, with some features, depending on how the question is phrased, and what type of function is asked for.

FEATURES ↳ Here we can define the main types of graph features:

- Maximum/Minimum values ↳ 'Turning points' of the graph: 
- Intercepts ↳ Where the graph crosses the x-axis or the y-axis.
- Line of symmetry ↳ Mainly regarding quadratics, self-explanatory: 
- Vertex ↳ Max. or min. points for quadratic function.
- Zeros, Roots ↳ Where $y=0$, i.e.: the x-intercepts.
- Asymptotes ↳ A line that a graph will get infinitely close to but will never touch, as the x or y tends to infinity.
- Intersection points ↳ The point where two lines cross (intersect).

FEATURES ON A GRAPH ↳ To visualise the above features:



GDC ↳ To graph almost any function, we press:

TI-nspire ↳ [B: GRAPH] ⟶ [TAB] ⟶ type function ⟶ [ENTER]

TI-84 ↳ [Y=] ⟶ type function ⟶ [GRAPH]

$\boxed{2.4}$ continued

FINDING FEATURES w/ GDC ↦ You can find the features previously defined by using your GDC's 'analyse graph' capabilities:

TI-nspire ↦ MENU → 6: ANALYSE GRAPH ... // TI-84 ↦ 2ND → CALC ...

FINDING FEATURES ↦ We can find many of these manually:

- Turning Points ↦ You can use calculus (see topic $\boxed{5}$), or for quadratics, you can use $x = \frac{-b}{2a}$ (see $\boxed{2.6}$ for A&A, $\boxed{2.5}$ for A&I), otherwise use a GDC.
- Intercepts ↦ To be on the y-axis, you must have $x=0$, so if you plug in $x=0$, and solve for y, you have the y-intercept. Reverse this for x-intercept(s).
- Zeros & Roots ↦ Again, set $y=0$, and solve.
- Vertical Asymptotes ↦ At an x value of restricted domain, such as the x value that means you would be dividing by 0.
- Horizontal Asymptotes ↦ To find what value the graph gets closer and closer to, we can plug in very large values for x. Or use advanced calculus (limits).
- Intersection Points ↦ See $\boxed{2.1}$ (set functions equal to each other, and solve)

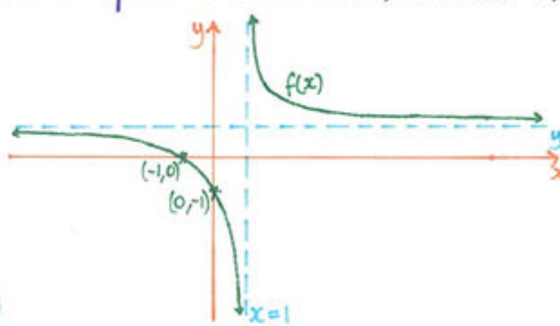
E.G. 1 ↦ Draw the graph of $f(x) = \frac{2}{x-1} + 1$:

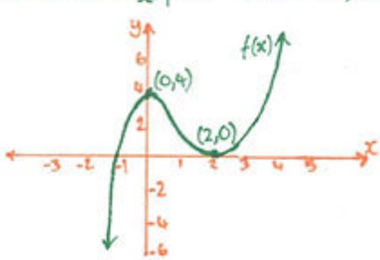
↳ Divide by zero when $x=1$, so V.A. at $x=1$

As $x \to \infty$, $f(x) = 0+1 = 1$, so H.A. at $x=1$

Plug in $x=0$: $f(0) = \frac{2}{-1} + 1 = -1$, y-int. at $(0,-1)$

Set $f(x)=0$: $0 = \frac{2}{x-1} + 1 \rightarrow x=-1$, x-int. at $(-1,0)$





E.G. 2 ↦ Draw the graph of $f(x) = x^3 - 3x^2 + 4$:

↳ y-int. at $f(0) = 0 - 0 + 4 = 4 \rightarrow$ y-int. at $(0,4)$

x-ints.: $x^3 - 3x^2 + 4 = 0 \rightarrow (x+1)(x-2)^2 = 0$, $x = -1$ & 2

Turning points: GDC, or calculus: $f'(x) = 3x^2 - 6x = 0$, $x = 0$ & 2

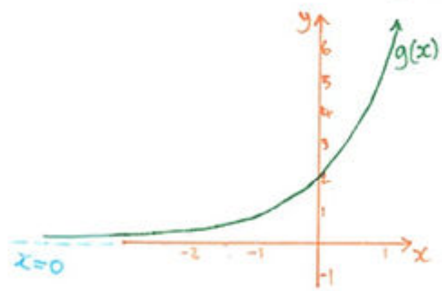
E.G. 3 ↦ Draw the graph of $g(x) = 2e^x$:

↳ Set $x=0$: $g(0) = 2e^0 = 2 \cdot 1 = 2$ ∴ y-int. at $(0,2)$

$0 = 2e^x$ has no real solutions, so no x-int.

As $x \to \infty$, $g(x) \to \infty$, but as $x \to -\infty$, $g(x) \to 0$

∴ H.A. at $x=0$



2.5 – Types of function

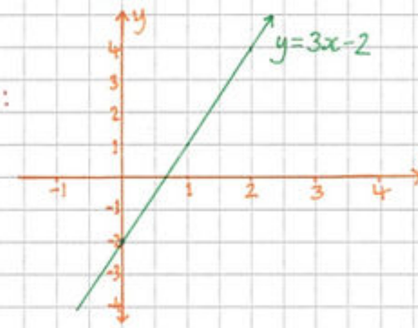
	SL	HL
A&A		
A&I	✓	✓

MODELLING ↦ We are going to cover many types of function in this section, so that we will have all the tools for the process of modelling. This means we will be able to assign a function to a real-world situation.

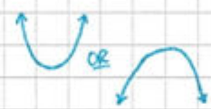
LINEAR FUNCTIONS ↦ This is in the form: $y=mx+c$ or $y=ax+b$. We will refer to the former here. This function is graphed as a straight line, with certain features:

FEATURES ↦
- y-intercept ↦ at $(0,c)$
- slope ↦ is 'm'

E.G. 1 ↦ Graph $y=3x-2$:
↳ y-int. at $(0,-2)$, slope of 3



QUADRATICS ↦ These are functions in the form: $f(x)=ax^2+bx+c$.
It takes one of the shapes shown on the right here: ⟶

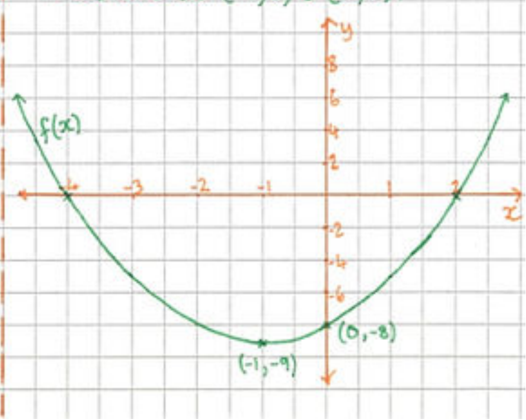


FEATURES ↦
- y-intercept ↦ at $(0,c)$
- Axis of symmetry ↦ at $x=\frac{-b}{2a}$
- Vertex ↦ at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$
- x-ints./roots/zeros ↦ can be found by:
  - By factoring
  - By quadratic formula: $\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$ In F.B.
  - By GDC, once function has been graphed:
    - TI-nspire ↦ [MENU] → [6: ANALYZE GRAPH] → [1: ZERO] → Choose lower & upper bound → [ENTER] → Root given
    - TI-84 ↦ [2nd] [CALC] → [2: ZERO] → Choose left & right bound → [ENTER] [ENTER]

E.G. 2 ↦ Graph $f(x)=x^2+2x-8$:
↳ y-int. at $(0,-8)$
- Axis of symm. at $x=\frac{-2}{2(1)}=-1$
- Vertex at $(-1, f(-1))$

$$f(-1)=(-1)^2+2(-1)-8$$
$$=1-2-8=-9$$

- By GDC or factoring: $(x+4)(x-2)$ roots are at $(-4,0)$ & $(2,0)$.

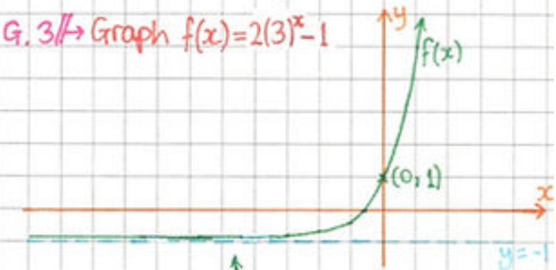


$\boxed{2.5}$ continued (2/3)

EXPONENTIALS → These are functions in the form: $f(x)=ka^x+c$. This is constantly increasing, as x (the exponent) increases. This exponent can often represent time (can be t, not x). This is relevant when a population might increase by a certain percentage each year – in that case, 't' would stand for years.

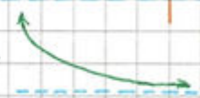
FEATURES →

E.G. 3 → Graph $f(x)=2(3)^x-1$



- y-int. → When $x=0$, i.e.: $ka^0+c=k+c$. At $(0,k+c)$
- Horizontal asymptote → At $y=c$
- Shape → For $k>0$ & $a>0$, always:

VARIATIONS → $f(x)=ka^{-x}+c$ has the same features, but this shape:

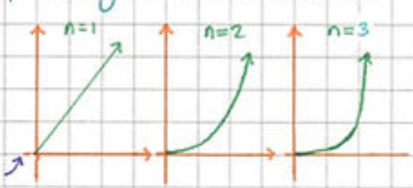


$f(x)=ke^{rx}+c$ also has the same features, but grows faster as r increases.

→ These functions are highly relevant to any kind of growth & decay, geometric sequences and more. We will see how to model these in $\boxed{2.6}$.

DIRECT VARIATION → This can be a linear, quadratic or other polynomial, but it only contains the highest degree term, such as $f(x)=2x$ or $V(r)=4\pi r^3$. The second example there was the volume of a sphere, but can be thought of as showing how volume *varies directly* as a result of changes in the radius.

- Takes the general form: $f(x)=ax^n$, $n\in\mathbb{Z}$
- It will always pass through the origin
- In the $x\geq 0$ region, it will have shape:

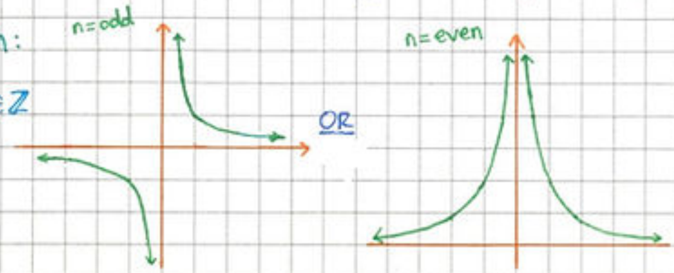


(steeper as n increases)

INVERSE VARIATION → You may have learned about *inverse proportion*, when two variables have the relationship: $y=\frac{k}{x}$, and you might see this with something like the relationship between length & width of a rectangle with a constant area. For example, if you have an area of 24cm, we know $w=\frac{24}{L}$. In IB, *inverse variation* extends to anything in the form:

$f(x)=\frac{k}{x^n}$, i.e.: $f(x)=kx^{-n}$, $n\in\mathbb{Z}$

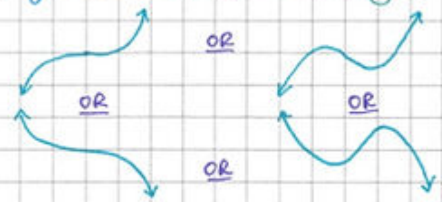
The graph of these functions take one of the forms:



2.5 continued (3/3)

CUBIC MODELS → These are functions in the form: $f(x) = ax^3 + bx^2 + cx + d$

You often just see these in the form $y = ax^3$ in A&I SL though. You won't need to know how to graph these in detail, but briefly: They will have the shape, such as →

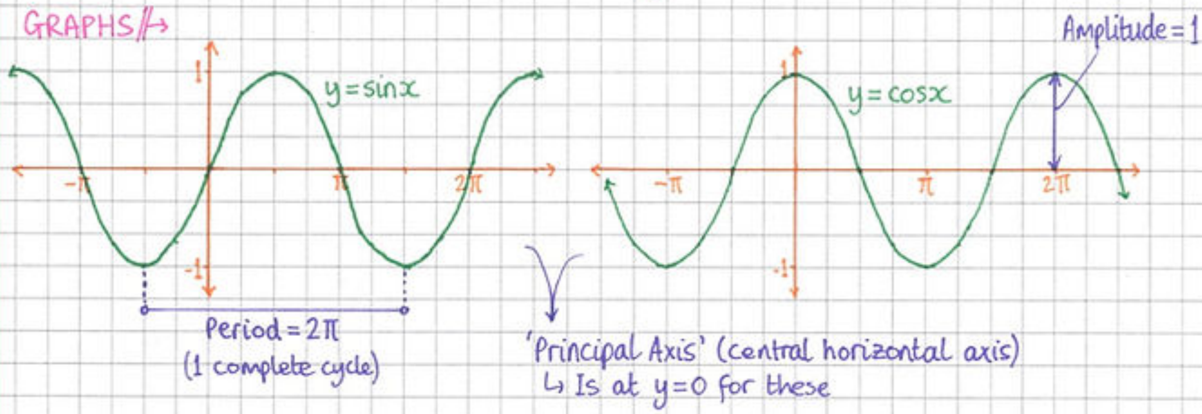


Other features include the fact that the y-intercept is at $(0, d)$.

→ You may see these used in many different fields, but volume is a common one. For example, volume of a sphere: $V = \frac{4}{3}\pi r^3$

SINUSOIDAL MODELS → This could easily be moved to your studies of Topic 3 if needed. But here is the basic knowledge for these models:

GRAPHS →



→ So they are defined by their *principal axis*, *amplitude* & *period* → which are $y = 0$, 1, and 2π respectively for both base functions. They differ with a horizontal shift, with $y = \sin x$ passing the origin $(0,0)$, and $y = \cos x$ passing $(0, 1)$.

→ Now we can apply transformations such as $af(x)$, $f(bx)$ & $f(x) + d$ to these functions. These stretch the waves vertically, horizontal, and shift vertically.

RULES →

$y = a\sin(bx) + d$	$\lvert a \rvert$ = amplitude $\frac{2\pi}{b}$ = period Principal axis at $y = d$
$y = a\cos(bx) + d$	same rules as above

E.G. 4 → $y = 2\sin\left(\frac{\pi}{2}x\right) - 3$:

↳ A sine curve with an amplitude of 2.

It has a principal axis at $y = 3$.

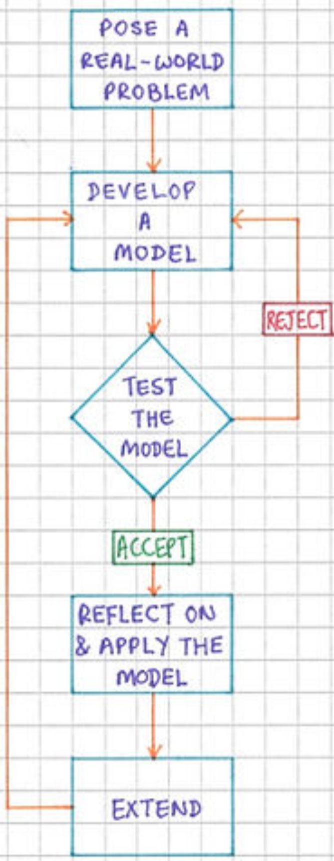
Period is $\frac{2\pi}{b} = \frac{2\pi}{\pi/2} = 4$

2.6 – Modelling

	SL	HL
A&A		
A&I	✓	✓

↦ *Modelling*, in this context, is a unique and very useful process, that allows us to truly understand a real-world problem. It involves finding a function (*model*) that fits and makes sense for the situation, and then using it to find out various information. It requires plenty of evaluation & critical reflection.

PROCESS ↦ This is an explanation of the process flow-chart that the IB gives for modelling.



↦ This will most likely be posed to you in an exam. You may pose your own problem in an IA (or in real life).

↦ There will usually be clues in the question for what type of model is needed. If something is *added periodically*, then it will be *linear*. Increasing by a % – it is *exponential*. *Cyclical?* Maybe *sinusoidal*. And so on... Find the correct constants in the question to insert into the model. Test it. Accept or reject as appropriate.

↦ Draw your graph, or plug something into your model to help actually answer the question at hand.

↦ The IB may give you follow-ups, extensions, or generalisations. It may need a new model, so return to step 2.

E.G. 1 ↦ Alfred joins a film streaming website. There are two options: either he can pay 'per film', where the 1st film costs \$1, then \$2.50 per film. A subscription model costs an initial \$5.50 to join, then \$3 per month. (a) Create a linear model for the 'per film' option: – Call a film 'f'. Take the first \$1, then add \$2.50 for every $(f-1)$ movies watched. $\therefore C = 1 + 2.5(f-1) \rightarrow \underline{C = 2.5f - 1.5}$ [for $f \geq 1$].

b) If he watches two films per month, how long until the other model is worth it?

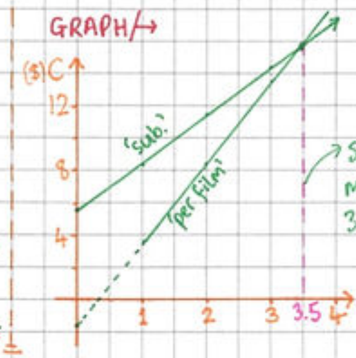
└ For 2 films per month, we have $m = 2f$. So now, 'per film' is $C = 3.5 + 5(m-1)$, so $C = 5m - 1.5$. 'Subscription' is $C = 5.5 + 3m$. See overpage for answer...

ALGEBRAICALLY ↦ We can solve this system to find the point the 'per film' model becomes more expensive

$$\left.\begin{array}{l} C=5m-1.5 \\ C=5.5+3m \end{array}\right\} 5m-1.5=5.5+3m \rightarrow 2m=7$$

$\rightarrow m=3.5$. i.e.: after 4mn.

GRAPH ↦



2.6 continued (2/3)

So 'subscription' starts off as more expensive, then after 3.5 (4) months, it's cheaper.

↳ This is common, depending on the situation, to *round up* for the final answer. Here, you only pay once per month, so we say 'sub.' is worth it after 4 months.

↦ *Linear models* are common, and the most straightforward. Many different questions could be asked about them though. E.G.: Interpret intercepts, slopes, evaluating, etc.

↦ For *quadratic models*, expect lots of projectiles, flying in ∩-shaped parabolas. Expect to have to work out when it 'hits the ground/sea' by finding the roots/zeros, or finding its vertex (max height), amongst other questions.

E.G.2 ↦ Uganda had a population of 44.5m on 1/1/19, and has a growth rate of ~4.9% per year. (a) Develop a model for Uganda's popⁿ over time:

↳ Call no. of years: 't' (from 1/1/19), popⁿ: 'p' (in millions). To increase something by 4.9%, you can multiply it by 1.049. ∴ we get $P=44.5(1.049)^t$

b) Estimate the popⁿ of Uganda on 1/1/2028:

↳ Plug in $t=9$: $P=44.5(1.049)^9 \approx 68.4 \rightarrow$ 68.4 million people

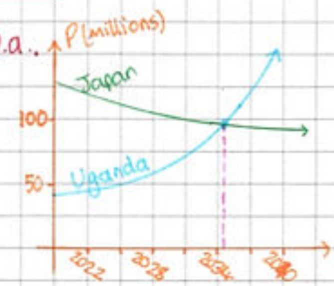
c) The Ugandan government has stated that due to its small area, they don't think they could sustain a population more than 95 million people. At current pace, what month would they pass this figure?

↳ Solve $44.5(1.049)^t=95 \rightarrow 1.049^t=2.13483... \rightarrow t=\log_{1.049}(2.13483..)$

$\rightarrow t=15.85347...$, $t=15$ yrs 10.24 mn $\rightarrow$ which is November 2034

d) Japan had 126.5m people in 2019, but is shrinking at 2.1% p.a.. When should Uganda's popⁿ overtake Japan's?

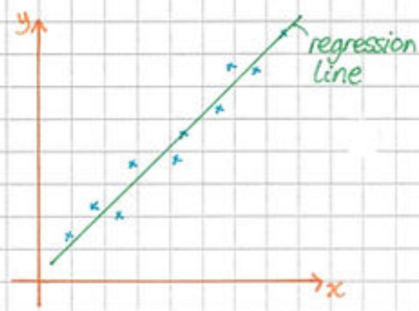
↳ Japan has model: $P=126.5(0.979)^t$. You can graph both on a GDC, then find int. point: $t=15.128$ yrs, i.e.: Feb. 2034



2.6 continued (3/3)

⊩→ For *trigonometric models*, the functions are derived from the x&y coordinates of *circles*. This means that a sin/cos function can perfectly model height as something moves in a circular path. In the past, the IB has given a few questions related to situations such as: height, moving around a ferris wheel. In that case, the length of a full cycle is *the period*, and the height from the low/high points to the centre – is *the amplitude*. Be prepared to solve trig. equations in order to find any remaining constants for your model.

REGRESSION ⊩→ You may have a situation where a model just *roughly* fit some measured data. This model (right) can be found yourself using technology. It is a linear model (a straight line), and this is a technique you will learn in Topic 4 → *linear regression*.

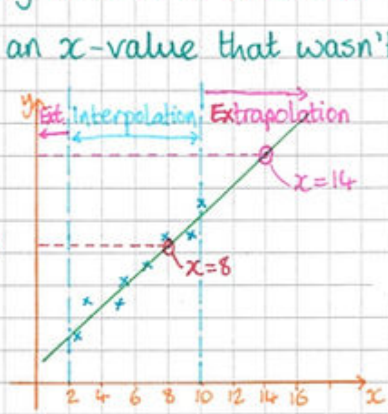


⊩→ You will also see these *regression models* for the various *non-linear functions*. For these, you should always be given the model in the question. You will take these models, use them, comment on them, or alter/extend them.

⊩→ Comments on usefulness of a model, or a specific prediction, may include:

- That time can't usually be negative
- For a projectile, it can't usually have a negative height
- You may realise that a different type of model might be more appropriate.

EXTRAPOLATION ⊩→ If you have used data from, say: $x=2$ to $x=10$, you may have a linear model that fits that data very well. You may be able to take that regression line equation and *estimate* a y-value for an x-value that wasn't in the data. It would be normal to estimate y for, say $x=8$ (as $2<8<10$). However, using the model to estimate for $x=14$ is called *extrapolation*. We can't know for sure what happens after $x=10$, so this practice is seen as less reliable/useful. However, it is often used in the real-world, with caution.



2.7 – Composite & Inverse [AHL]

	SL	HL
A&A	2.5	2.5
A&I		✓

COMPOSITE FUNCTIONS → Instead of just plugging in a *number* into a function, as x, in a *composite function*, we input an entire function into another one. This makes a whole new function in x.

NOTATION → $(f\circ g)(x)$ means that you are inserting $g(x)$ into $f(x)$. In other words:

$$\boxed{(f\circ g)(x)=f(g(x))}$$

You can view this as one, new, function. Or you may see it as a two-step process that runs x through the 'g' function, then runs that result through the 'f' function.

E.G.1 → If $f(x)=2x+1$ & $g(x)=3-4x$, find $(f\circ g)(x)$:

→ $f\circ g(x)=f(g(x))=f(3-4x)=2(3-4x)+1=6-8x+1=\underline{7-8x}$

→ We can also *evaluate* a composite function as well, inputting a number:

E.G.2 → If $f(x)=6x-5$ & $g(x)=x^2-x$, find $(g\circ f)(1)$:

→ Either find $(g\circ f)(x)$ and plug in 1 OR just do $f(1)$, then $g(f(1))$:

Method ① $(g\circ f)(x)=(6x-5)^2-(6x-5)=36x^2-60x+25-6x+5 \rightarrow 36(1)^2-60(1)+25-6(1)+5=\underline{0}$

OR

Method ② $f(1)=1$, $g(f(1))=g(1)=1^2-1=\underline{0}$

INVERSE → Finding an inverse function was explained in 2.2 already, but in HL, we need to see how they interact with composite functions, as well as seeing when a function may not have an inverse.

E.G.3 → Take $g(x)=\sqrt{x+1}$ & $h(x)=5x-3$, find $(h\circ g^{-1})(x)$:

→ Firstly, find $g^{-1}(x)$: $y=\sqrt{x+1} \rightarrow x=\sqrt{y+1} \rightarrow x^2=y+1 \rightarrow g^{-1}(x)=x^2-1$

Then, $h(g^{-1}(x))=5(x^2-1)-3=5x^2-5-3=\underline{5x^2-8}$

CHECKING INVERSES → If you run a number through a function, then plug that result into the *inverse*, then by definition, it should return you to the original number, i.e.:

$$\boxed{(f\circ f^{-1})(x)=x}$$

DOMAIN RESTRICTION → Reminder: If a function is *one-to-many*, then it is not technically a function, but just a *relation*. Given that an inverse essentially switches x & y, it also can switch a valid many-to-one function (e.g.: $y=x^2$) to an invalid function (e.g.: $y=\sqrt{x}$). You may still find the inverse, but you can *'restrict the domain'*, to force it to be one-to-one. For example, $f(x)=x^2 \rightarrow f^{-1}(x)=\sqrt{x}$ but for $x\geq 0$ only.

E.G.4 → Find the inverse of $f(x)=x^2+2x-3$, by restricting the domain:

→ Take $f(x)=x^2+2x-3 \xrightarrow[\text{the square}]{\text{complete}} f(x)=(x+1)^2-4 \xrightarrow[x\,\&\,y]{\text{switch}} x=(y+1)^2-4 \rightarrow x+4=(y+1)^2 \rightarrow \sqrt{x+4}=y+1 \rightarrow y=-1+\sqrt{x+4} \rightarrow f^{-1}(x)=\underline{-1+\sqrt{x+4}}$. So we could say $f(x)$ 'does not have an inverse'. But for $\underline{x\geq -4}$, then f^{-1} is a function.

2.8 – Transformations [AHL]

	SL	HL
A&A	2.11	2.11
A&I		✓

THEORY// ↦ There will be six types of transformations from $f(x)$:

- → $y=f(x)+b$ ↦ you are adding 'b' to every y-coordinate
- → $y=f(x-a)$ ↦ you take the y-coord. from an original x that was 'a' units to the left
- → $y=pf(x)$ ↦ you multiply every y-coordinate by 'p'
- → $y=f(qx)$ ↦ you take the y-coord. from the 'qx' input, and apply it to x instead
- → $y=-f(x)$ ↦ you change the sign of every y-coordinate
- → $y=f(-x)$ ↦ you take the y-coord. from the '$-x$' value of x

RULES// ↦ Therefore, these then correspond to the following rules:

- → $y=f(x)+b$ ⇒ Shift up by 'b' units
- → $y=f(x-a)$ ⇒ Shift right by 'a' units

} TRANSLATIONS

- → $y=pf(x)$ ⇒ Stretch vertically by scale factor 'p'
- → $y=f(qx)$ ⇒ Stretch horizontally by S.F. '$\frac{1}{q}$'

} STRETCHES

- → $y=-f(x)$ ⇒ Reflect in the x-axis
- → $y=f(-x)$ ⇒ Reflect in the y-axis

} REFLECTIONS

E.G.1// ↦ Describe the transformation from $y=x^2$ to $y=(x+8)^2$:

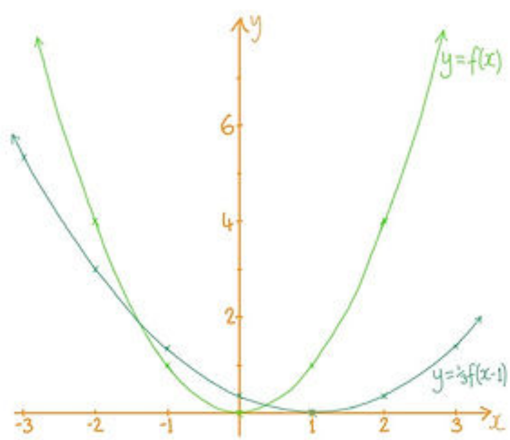
↳ Translation to the left by 8 units

E.G.2// ↦ Stretch $y=\cos x$ vertically by S.F.=2 & reflect in the y-axis:

↳ $y=2\cos(-x)$

E.G.3// ↦ If $f(x)=x^2$, graph $y=f(x)$ & $y=\frac{1}{3}f(x-1)$:

↳ We shift $f(x)$ to the right by 1. Then, Multiplying by ⅓ will mean that the original graph is stretched vertically by a scale factor of ⅓, giving ⇒



ORDER// ↦ The order of transformations is *significant*. They essentially follow the *order of operations*.

E.G.4// ↦ Describe the geometric transformations needed to go from $f(x)=x^3$ to $g(x)=6-3(x+2)^3$:

↳ By the order of operations: ① horiz. translation, 2 units to the left // ② vert. stretch, S.F.=2 // ③ Reflect in the x-axis // ④ Vertical translation, 5 units up.

2.9 – Further Modelling [AHL]

	SL	HL
A&A		
A&I		✓

MODELLING ⊩→ Modelling was covered extensively in 2.6, and various types of functions that we use for modelling in A&I SL was seen in 2.5. These were: *linear*, *quadratic*, *exponential*, *direct/inverse variation*, *cubic*, and *sinusoidal models*. In this section, we won't go over the basics of the modelling process again, but simply introduce a few extra types of function/model. These will be: *natural logarithmic*, *further sinusoidal*, *logistic*, and *piecewise models*.

HALF-LIFE ⊩→ Exponentials were covered in 2.5, as mentioned, but HL students may need to be familiar with one extra concept, which is *half-life*. This is just one detail often included when talking about *radioactive decay* – which is exponential. If we have substance that is losing a certain percentage of its mass every time a certain time period passes, then we can find the half-life: the time taken to lose *half its mass*.

E.G. 1 ⊩→ 1200 mg of Fermium-253 decays to 119.055 mg over the course of 10 days. Find its half-life:

↳ If y is mass, and t is time in days, then $y=A(B)^t$. When $t=0$, $y=1200$, so $1200=A(B)^0 \rightarrow A=1200$. Then, when $t=10$, $y=119.055 \rightarrow 119.055=1200(B)^{10} \rightarrow 0.099213=B^{10} \rightarrow B=\sqrt[10]{0.099213}=0.7937$. So we have $y=1200(0.7937)^t$

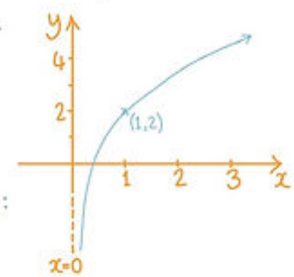
Setting $600=1200(0.7937)^t \rightarrow 0.5=(0.7937)^t \rightarrow \log_{0.7937}0.5=\underline{3.00 \text{ days}}$

NATURAL LOG MODELS ⊩→ These will be models in the form: $f(x)=a+b\ln x$. Reminder: $\ln x$ is just the same as $\log_e x$, where 'e' is an important irrational constant in mathematics, where $e \approx 2.71\ldots$

The graph of this function grows fast at first, then 'tails off', albeit continuing to increase as $x \rightarrow \infty$.

→ Vert. asymptote at $x=0$
↳ Passes through $(1, a)$ as $\ln 1=0$.

For example, this is the graph of $y=2+3\ln x$:



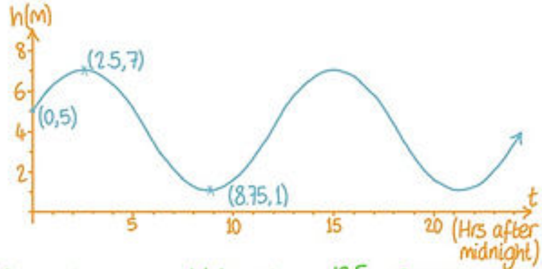
⊩→ This type of model is rarely used by itself in exams, and by its nature, you would never be expected to come up with a natural log model from a worded explanation of the situation. You would therefore be given the model, and asked follow-up questions. It is more likely that you'd see this when you have linear model for data on a y vs. $\ln x$ graph → more on this in 2.10. Examples may include earthquake intensity, box office gross, etc.

MORE SINUSOIDAL MODELS ⊩→ Again, we just have a small extension on these for the HL section. Where before we just saw $f(x)=a\sin(bx)+d$, now we have $f(x)=a\sin(b(x-c))+d$. Looking at the previous section, the extra '$-c$' is equivalent to the transformation listed as $y=f(x-a)$, which was a shift to the right. In sinusoidal functions, we'll call it a *phase shift*, which is still simply a shift left/right by a certain *angle*. Here, it becomes extra important to know how the base sin/cos functions 'start': i.e.: $\sin(0)=0$ [and rising] & $\cos(0)=1$.

REGRESSION ⊩→ As with the majority of these types of functions, you may have *estimate* a model using non-linear regression. In this case – the *sinusoidal regression* option on your GDC.

2.9 continued

E.G.2 ⟼ The following graph shows the height (m) of the water at Skegness Pier over a day. Find a function for this in the form $h(t)=a\sin(b(t-c))+d$:



↳ Amplitude: $a=\frac{7-1}{2}=3$. Period $=2(8.75-2.5)=12.5$

$b=\frac{2\pi}{\text{period}}=\frac{2\pi}{12.5}$. $d=\frac{7+1}{2}=4$. With no phase shift, the first maximum would be at $x=\frac{12.5}{4}=3.125$, as it's quarter of a period. So it has been shifted 0.625 left. $\therefore h(t)=\underline{3\sin(\frac{4\pi}{25}(x+0.625))+4}$

LOGISTIC MODELS ⟼ This will usually be a completely new type of function for most students. To get to the point, these are functions in the form: $\boxed{f(x)=\frac{L}{1+Ce^{-kx}}}$ where $C,k>0$

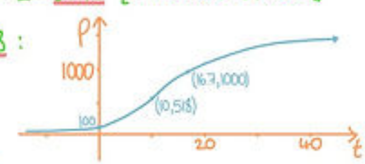
It will produce a graph in the form of an S-curve, or *sigmoid curve*: So anything that grows quickly initially, then flattens off, for example, population, or amount of bacteria, can be modelled with a logistic curve.

- ↦ Horizontal asymptote ('limiting value') at $y=d$, and technically at $y=0$ to the left
- ↦ y-intercept: plug in $x=0$, which always gives $(0,\frac{L}{1+C})$
- ↳ Rotational symmetry around the point $(\frac{\ln C}{k},\frac{L}{2})$, which is an *inflection point*, see topic *5.10*.

E.G.3 ⟼ After millions of years without any new mammal species in New Zealand, a couple thousand rats arrived unknowingly on Tasman's ship in 1642. They realised it was a problem 20 years later, so they surveyed: they estimated there were 100 thousand rats in NZ. After 10 more years, there were 518k. If 't' is time in yrs after 1662, and P is pop^n of rats in thousands, and it is known that rat pop^n's follow a logistic curve of: $P(t)=\frac{L}{1+Ce^{-0.2t}}$

(a) Find C & L: We have (0,100) & (10,518). Plug both in to get: ① $100=\frac{L}{1+Ce^{0}}=\frac{L}{1+C}\rightarrow L=100+100C$ // ② $518=\frac{L}{1+Ce^{-2}}$ $\rightarrow L=518+\frac{518}{e^{2}}\cdot C$ // Sub: $100+100C=518+\frac{518}{e^{2}}\cdot C\rightarrow C=\underline{13.98}$ // Sub into ①: $100=\frac{L}{1+13.98}\rightarrow L\approx\underline{1498}$ [∴ max rats = 1498165]

(b) When did the pop^n surpass 1 million? $P(t)=\frac{1498}{1+13.98e^{-0.2t}}=1000\rightarrow t=16.7$ ∴ in $\underline{1678}$:



PIECEWISE MODELS ⟼ Across a wide variety of real-world situations, it may be more appropriate to use one model for some of the time, but then have a different model for a section of the domain. For example, if a bird flies normally, then dives in to water to catch a fish, then the model might be linear for some time, then quadratic:

E.G.4 ⟼ If \$24,000 earns 2% simple interest per yr. for 5 yrs, then 3.8% compound int. after that, (a) give the model:

↳ Simple: $F=24000(1+0.02n)$. For compound, we need 5yr value, $24000\times(1.1)=26{,}400$. So, $F=26400(1.038)^{n-5}$, overall, giving us:

$$F(n)=\begin{cases}24000(1+0.02n), & 0\le n\le 5\\ 26400\cdot(1.038)^{n-5}, & 5<n\end{cases}$$

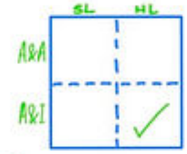
E.G.5 ⟼ Linh is running a 10k. It takes her a minute to even cross the start line. Then she starts very fast, but can't keep it up, and starts to slow down. After 20mins she decides to just go at a constant pace until the end. So: $s(t)=\begin{cases}C\ln t, & 1\le t<20\\ 2+0.15t, & t\ge 20\end{cases}$

s is distance in km, t is time in mins.

(a) Find C: $2+0.15(20)=5\text{km}\twoheadrightarrow 5=C\ln(20)\rightarrow C=\frac{5}{\ln 20}\approx\underline{1.669}$

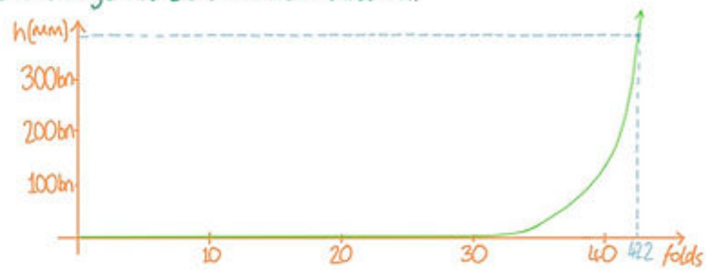
(b) Does she get under an hour? $10=2+0.15t\rightarrow 8=0.15t\rightarrow t=53.3\text{ min}<1\text{hr}$. ∴ $\underline{\text{Yes}}$

2.10 – Log Graphs & Linearising [AHL]



LOG GRAPHS → There are certain situations where the scale of the numbers simply are tricky to observe properly on a regular graph, because such small and large numbers are both involved.

E.G.1 → A piece of paper is 0.075mm. We fold it repeatedly (beyond the usual limit of 7). Use a graph to find when it becomes 1cm thick, then how many folds until it reaches the moon?
↳ We graph $h(n)=0.075(2)^n$: Moon ≈ <u>42</u> folds

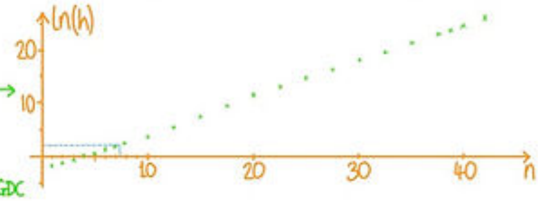


→ But we never found the 1cm n-value. That because 10mm is so ridiculously small on a scale of 100's of billions of mm's. If we zoomed to a cm, we wouldn't see the 'moon-height'. How do we fix this issue? Well, the issue is that doubling is *powerful*. So we reverse that with *logs*. Specifically, we graph $\ln(y)$ vs. x instead of the regular y vs. x.

E.G.1b → Graph the height of the paper against no. of folds on a $\ln(h)$ against n graph:
↳ We need to take each h, and do $\ln(h)$ ↗

n	h	ln(h)
1	0.15	-1.897
2	0.3	-1.204
3	0.6	-0.51
⋮	⋮	⋮
20	78643.2	11.27
⋮	⋮	⋮
42	329.8bn	26.52

↳ via GDC →



→ Here, we clearly see one of the most helpful features of these *'semi-log'* graph, that we can easily see the 1cm scale with the moon scale. In fact, as 1cm = 10mm, if we do $\ln(10)=2.30$ and draw a line across -----, we can finally answer Q via graphing: we see it passes 1cm between 7th & 8th folds.

EXPONENTIALS → This process can be applied to any exponential function in the form: $y=a(b)^x$. Why is this? Well, if you do $\ln y$, you get $\ln y=\ln(a(b)^x) \rightarrow \ln y=\ln a + x\cdot\ln b$, and as $\ln a$ & $\ln b$ are just constants, then this is a version of $mx+c$. i.e.: a linear function, but only on the semi-log graph.

REGRESSION → Given you know how to find linear regression lines by now, you apply that to one of these graphs, to get what is effectively an *exponential regression curve*. Just use the $\ln y$ & x as your 'y' & 'x' lists on your GDC:

E.G.2 → For the following data, showing mass of bacteria (M, gr) after t hours, show that it should fit an exponential model, giving the equation for M in terms of t:

t(hrs)	0	1	2	3	4	5	6	8	10
M(grams)	2.7	3.3	4.2	5.0	6.4	7.5	8.9	14.2	21.7

→ On GDC, we find a lnM column:

lnM	0.993	1.194	1.435	1.609	1.856	2.015	2.186	2.653	3.077

→ Semi-log graph:

→ Very close to linear, so original data is exponential. Using GDC linreg(mx+b) function with x list = t & y list = lnM. $r^2=0.999$ which confirms exp. correlation.

GDC also shows $m=0.2065$ & $b=0.9964$, which means $\ln M=0.2065t+0.9964 \rightarrow M=e^{0.2065t+0.9964}$

$M=e^{0.2065t}\cdot e^{0.9964} \longrightarrow M=2.7084(e^{0.2065})^t \longrightarrow \underline{M=2.71(1.23)^t}$

→ Which shows it can be modelled with an exp. model. This one growing by ~23% per hour

2.10 continued

POWER MODELS ↦ As discussed in 2.5, we may have work with models in the form $f(x)=ax^n$. However, now we know about switching to axes for exponentials, we investigate something similar for these power models:

Take $y=ax^n$, find log of both sides: $\ln y=\ln(ax^n) \to \ln y=\ln a+\ln x^n \to \ln y=\ln a+n(\ln x)$, which again, is almost a linear function. Although this time, not only do we switch y for lny, but also switch x out for lnx as well. In practice, this actually means we can graph it with *two log axes*, to *linearise* the data.

E.G.3 ↦ A group of students try to model the motion of a football being dropped off their HS roof terrace:

DATA RECORDED:

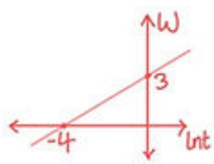
time (s)	dist. fallen (m)
0.5	1.3
1	5
1.5	11
2	19
2.5	29
3	43
3.5	62.5

⇒

So we get our ln(f) & ln(t) columns on our GDC, and find the linreg(mx+b) line for those. We get m=1.966 & b=1.608 this actually means: $\ln f=1.966(\ln t)+1.608 \to f=e^{1.966(\ln t)+1.608} \to f=e^{1.966(\ln t)}e^{1.608} \to \underline{f=4.99t^{1.97}}$ [a 'power model']

LOGARITHMIC MODELS ↦ These were seen in 2.9, but now we'll see how *log axes* can be used with this type of function, and how it can be *linearised*. So we saw how exponentials can be written in the form $\ln y=mx+b$, and how power models can be written in the form $\ln y=m(\ln x)+b$. You can imagine that there is one other version of this – where we just have $\ln x$, which subsequently means that we can change the x-axis to an $\ln x$ axis in order to linearise it. If we take a look at the general form of the natural log function introduced before: $y=a+b\ln x$, we see no work is needed to show that this is the function we need.

E.G.4 ↦ Give a natural logarithmic function that fits the following graph:

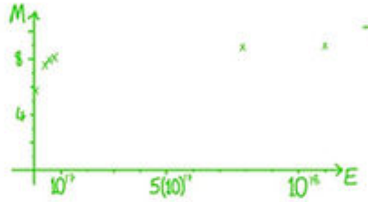


→ As we have a straight line, we can start by finding the slope: $m=\frac{3-0}{0--4}=\frac{3}{4}$, W-intercept is 3, therefore we have $\underline{W=3+\frac{3}{4}\ln(t)}$

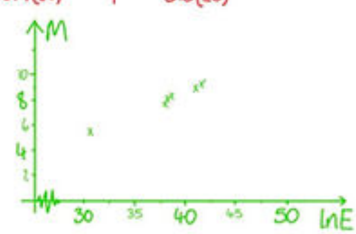
E.G.5 ↦ Find an appropriate function to model earthquake magnitude vs. energy released (Joules), given this data:

EARTHQUAKE	San Fran. (1906)	Morocco (1960)	Sumatra (2004)	Japan (2011)	Nepal (2015)	Turkey (2023)
MAGNITUDE	8.2	5.8	9.1	9.0	8.1	7.8
ENERGY REL.	$5.0(10)^{16}$	$2.0(10)^{13}$	$1.1(10)^{18}$	$7.9(10)^{17}$	$3.9(10)^{16}$	$3.0(10)^{16}$

↳ Try a scatter plot:



→ Tricky to see, but definitely closest to the logarithmic shape, so we find lnE, and plot M vs. lnE:



We have a linear relationship between M & lnE [r=0.994]. GDC gives m=0.30285 & b=-3.5067, so:

$\underline{M=-3.51+0.303\ln E}$ → which shows that as you move 1 up the Richter scale, energy increases 27x

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↦ So apart from using log graphs to find the model, you can just use the graphs to show large & small numbers.